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George B. Dantzig, Patrick H. McAllister and John C. Stone[1]

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ABSTRACT

Given a general dynamic equilibrium formulation of a time staged model, ~~we~~ seek conditions on the distribution of utility functions of individuals which imply the model is equivalent to a mathematical program.

Gorman and others long ago have observed that Engel curves of average consumption as a function of income at fixed prices are remarkably linear over a broad range of income of interest which tapers off at both ends of this range. ~~We~~ reproduce this phenomenon by assuming ~~(a)~~ that a general polynomial of the second degree has enough parameters (coefficients) to globally represent the utility functions of individual consumers, and (b) the distribution of utility functions that individuals have is independent of the income they happen to have. ~~We achieve the latter by~~ assigning values to the parameters of the utility functions by a random drawing with replacement from a "population urn" containing a representative sets of the parameters. ~~We~~ then derive the functional form of the per capita demand function and necessary and sufficient conditions for its integrability.

Finally, ~~we~~ show, in the context of the time staged model, that when the population is not too polarized as to its tastes at fixed income levels, a concave objective function always exists, which maximized subject to the physical flow constraints, implies the equilibrium conditions.

Introduction

Given a general dynamic equilibrium model, a long standing problem is finding conditions on the distribution of utility functions of individuals which guarantee that the model is equivalent to the problem of maximizing a concave objective function of the aggregate consumption variables subject to the physical constraints of the system. When the two are equivalent, the powerful software of mathematical programming can be applied to efficiently solve large scale equilibrium problems.

Our approach differs from past ones by the way we assign values to the set of a parameters of the utility function of individuals. We first assume the distribution is independent of the income individuals happen to have. We achieve this independence by placing representative vectors of parameters values in an "urn" and assigning them to each individual at an income level by a random drawing with replacement from the urn.

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In the context of a time-staged equilibrium model, we are therefore not interested in deriving the functional form of the per capita demand function of some future time period. since it is now a random function, but we are interested in **expected** demand as a function of prices and per capita income. By the law of large numbers, the expected per capita demand will differ from actual future per capita demand insignificantly for a population the size of the U.S. Accordingly we will be seeking conditions on the distribution of representative utility function parameters in the urn which imply that these expected per capita demand functions are integrable.

An early controversy which arose in theoretical work on consumer demand concerned the shape of Engel curves which express at fixed prices average consumption of a particular category of goods (such as food) as a function of income level. Survey data strongly suggest that these curves are nearly linear over a broad range of income levels, see Graphs 1-8 at the end of this paper. Some authors have conjectured that his must be true because the underlying demand functions of individuals must be linear, or nearly so, at fixed prices over a broad range of individual income available for consumption.

For example, Gorman in his 1953 paper [9] remarks: "A great deal of work has been done on Engel curves particularly by Allen and Bowley and Houthakker. The work of Allen and Bowley was based on the assumption that the classical Engel curves for different individuals at the same prices were parallel straight lines, but this has been rejected in work of Houthakker in favor of a doubly logarithmic form. However, the earlier assumption fits the data remarkably well." [22,23].

Gorman, in the article cited above, finally showed conclusively that average linear demand could only happen if the underlying individual demand functions are linear in income at fixed prices.

Since linearity of individual demand functions in income at fixed prices was considered unlikely for theoretical reasons, researchers have concentrated on using the logarithmic functional form following the traditional approach to demand analysis found in the pioneering work of Henry Shuitz, Richard Stone, and Herman Wold, [24,25,26].

The logarithmic utility functions were generalized by Christianson, Jorgenson, and Lau [28,30] who proposed that utility be approximated by a function which is quadratic in the logs of the consumption variables. Certain difficulties in translating from individual demand to aggregate demand which were present in their early work were overcome by a more general theory which the authors Jorgenson Lau and Stoker call "exact aggregation", see [11]. They consider a model in which each individual has a trans-log utility function which depends on "attributes" of the individual as well as consumption itself.

By placing restrictions on the way that these attributes enter into the utility function, they were able to find a model in which demand exactly aggregates thereby deriving per capita and aggregate demand as a function of prices and a certain class of symmetric statistics of the income distribution, which includes more than just the mean or total aggregate income. In this way they

are able to obtain an exact aggregation result without appealing to linearity in individual or average demand at various income levels.

Our approach is motivated theoretically by the observation that when the utility functions of individuals are polynomial expressions of the second degree in the consumption variables, the demand function of each individual will be linear over a broad range of income at fixed prices tapering off to zero at the poverty end as more and more individuals, in order to maximize their utility, have to set more and more components of their consumption vector to zero, and leveling off at the high income end as more and more individuals have income sufficient to purchase their "satiation" vector. Consistent with empirical observation, this implies that **expected** demand functions (Engel curves) are linear over a broad range of income (used for consumption), and this in turn implies that the **expected** per capita demand is linear in per capita income over a broad range of per capita income.

There is no obvious reason why these steps should lead to an expected per capita demand function for a given time period t that is integrable; or if integrable within a period why this should imply, in the context of a time-staged equilibrium model, that one can replace the dual price constraints by an objective function which can be maximized subject to just the primal system of physical-flow constraints to obtain the equilibrium solution. Indeed from Arrow's Impossibility Theorem, we know that seeking an aggregate utility function for the economy over time could be a futile quest since in *general it need not exist* [1, 5, 18, 27].

Conditions for integrability of a demand function have been given by Slutsky, see Varian [19]. We derive a necessary and sufficient condition for integrability based on the derived form of the expected aggregate demand function for period t.

It all depends on how the parameters of the utility functions of individuals in urn are distributed. One measure of how much utility functions U^i differ from one another is to compare H^i, where H^i is the price cross-effect matrix of the i-th utility function in the urn with $\bar{H}$ any positive definite matrix used for comparison (such as average H^i) by forming $\bar{H}^i = \bar{H}^{-1/2} H^i H^{-1/2}$. We prove, in a worse case scenario, that if $\rho_i < 3 + \sqrt{8} \doteq 5.83$ for every i, where ρ_i, is the ratio of the largest to smallest eigenvalue of $\bar{H}^i$, then the expected per capita demand function is integrable. Moreover, the greater the variability of orientations of the axes of the ellipsoids $p' \bar{H}^i p = constant$ the higher is the bound for ρ_i.

Outline of the Paper

In Part I, we derive the form of the expected per-capita demand function for each discrete time period t; in Parts II and III we derive necessary and sufficient conditions for its integrability; in Part IV, we present the time-staged dynamic equilibrium model for consumers and producers/investors. In Part V, we develop a first-order approximation to the expected aggregate demand and utility functions for a period which allows one to explicitly express these functions and estimate their parameters. Finally we estimate the parameters of the expected per capita demand function and

the per capita utility function using survey data and test the theory empirically by using it to predict time-series data of per capita consumption of various items when prices and per capita income are given.

Even when the latter exists, if the numeraire for normalizing period t prices is not suitably chosen, we prove in the context of time-staged equilibrium model that a utility function that drives the economy over time need not exist. However, when a numeraire is suitably chosen, the dynamic equilibrium model is integrable.

Prices normalized by this numeraire differ very little in practice from those obtained by scaling them so that their average price is unity, see last column of Table 4. Thus, from the viewpoint of the investor, these prices are equally acceptable for calculating the rates of return of various investment possibilities. If it is, then it is possible to restate the dynamic equilibrium problem as a mathematical program and to use non-linear programming software like MINOS to optimize the primal system [15].

To test the theory, fits were made to survey data (see Table I) and tested by predicting the consumption pattern of final consumers for the years 1961 to 1982 as a function of prices and per capita income used for consumption (see Graphs 9 to 16). We also report on an experiment that suggests the approximation may be a very good one even if the set of utility functions U^i in the urn differ markedly from one another.

Motivation of our Research

Our presentation here arose out of our efforts beginning in 1975 to build a macro-economic model of U.S. to assess the long- term effects of modernization, innovation, foreign competition, energy prices, and conservation on the growth of various economic sectors, GNP, and per capita income. PILOT, is a multi-time period model, quite large, with a data base of over 70,000 technological coefficients, [4]. Its principal weakness, as we see it, lies not with the numerical results from various scenario runs (the physical growth of the economy appears to be quite reasonable), but with our inability to justify the aggregate utility function which we had devised to increase the standard of living and had been using as the driver as though the U.S. were a planned economy. To be more precise, the partial derivatives of this somewhat arbitrarily chosen utility function, *interpreted as prices*, implied a behavioral response of final consumers to prices and a behavioral response of producers/investors also to prices, which are almost certain to be out of kilter with what their observed behaviors would be in the real world.

These considerations led us recently to reverse the process and to reformulate PILOT along classical economic lines as a dynamic equilibrium model that satisfies the behavioral responses of final consumers to prices given their income for consumption, and the responses of producers/investors in choosing activities that yield at least a minimum rate of return. From a mathematical point of view, this is not a dramatic change, since the set of Kuhn-Tucker conditions [12]

that must be satisfied at the point which maximizes the utility function, are quite analogous to the Arrow-Debreu conditions [1,5] that must be satisfied at an equilibrium point. The main difference is that the general equilibrium problem belongs mathematically to the more general class of complementarity problems which require for solution combining the dual system of price constraints along with the primal system of physical constraints in one big simultaneous system, [3, 6, 8, 13, 17]. For a problem the size of our PILOT model, this combined system is too large to solve directly and much of our research has been concerned with finding efficient ways to use mathematical programming software, like MINOS [15], to solve the system, including ways to decompose the system into smaller problems and to use their solutions to iteratively converge to an equilibrium.

A fundamental question that has concerned us is reconciling the "prescriptive" (normative) view of the initial formulation with the "descriptive" (behavioral) view of the dynamic equilibrium formulation. To be precise, do the realistic behavioral assumptions of an equilibrium model serve as a driver promoting reasonable growth and well being of the economy when it has the potential for such growth? We will see later that the dynamic equilibrium formulation implies under reasonable conditions an objective for the economy the form of which makes it evident why growth in the economy will occur if it has potential for growth.

PART I:
DERIVING THE PER CAPITA DEMAND FUNCTION FOR PERIOD t

Notation

For discussion involving a fixed period, we will usually omit the time subscript. Thus the consumption vector of the j-th consumer in period t is denoted by $X^j = X_t^j$; his *budget* or personal income for consumption is denoted by $I = I_t$ measured in period t undiscounted dollars. Expected *per capita* consumption, income, and utility are denoted by $\bar{X}_t, \bar{I}_t, \bar{U}_t$ and the corresponding *aggregates* by bold face $\mathbf{X}_t = P_t \cdot \bar{X}_t$, $\mathbf{I}_t = P_t \cdot \bar{I}_t$, $\mathbf{U}_t = P_t \cdot \bar{U}_t$ where P_t is the size of the population in period t.

We will use the symbol $\mathcal{I}$ to denote the identity matrix to avoid confusion with I and $\bar{I}$ which refer to income. The inner product of a column vector v with itself will be denoted by $v^2 = v'v$. The symbols $\alpha, \beta, \delta, \lambda$ denote scalar constants. L.H.S and R.H.S. are abbreviations for *left hand side* and *right hand side* of an equation or an inequality relation. The symbol $\doteq$ means *approximately equal.*

Utility Function of Individuals

The first assumption we make is that each final consumer has a utility function that is quadratic in the consumption variables and that he chooses his consumption vector by maximizing his utility function subject to his budget constraint. This specific functional form may be viewed as a second-order global approximation to whatever may be his true utility function.

Assumption 1. *Individual j in period t has a utility function $U^j(X)$, measuring the value j attaches to having a consumption vector X, which can be represented by a general quadratic function of the form*

$$U^j(X) = 2(M^j S^j)'X - (X)'M^j(X) + Constant_j \qquad (1.1.0)$$

where vector $S^j > 0$ and matrix M^j is symmetric and positive definite, hence non-singular.

Without loss of generality, we may rescale the matrices M^j so that, letting $e' = (1, 1, \ldots, 1)$, their inverses H^j have the property

$$e'H^j e = \sum_k \sum_\ell H^j(k, \ell) = 1 \,, \quad H^j = (M^j)^{-1} \,. \qquad (1.1.1)$$

Letting $Constant_j = -(S^j)'M^j S^j$,

$$U^j(X) = -(S^j - X)'M^j(S^j - X) < 0 \,, \quad \text{for all } S^j - X \neq 0. \qquad (1.1.2)$$

It is easy to see $U^j(X)$ in (1.1.2) is unconditionally maximized when $X = S^j$. Therefore, it is natural to assume that S is strictly positive and to refer to S^j as the "satiation" vector of individual j. We

view the *income* I of an individual as an "authorization" to expend up to that amount for actual consumption. Should it happen that an individual's budget $I \geq p'S^j$, he maximizes his utility by buying his satiation vector S^j. The unexpended amount $I - p'S^j$, in this case, is not used.

Form of the individual demand function. The budget constraint and nonnegativity constraint for individual j in period t at fixed prices p are:

$$p'X \leq I, \qquad X \geq 0, \qquad p > 0. \tag{1.2.0}$$

Subject to (1.2.0), j maximizes his utility function $U^j(X)$. The vector X which maximizes $U^j(X)$ subject to a budget constraint is denoted by X^j. This is a special case of a quadratic programming problem [3]. We distinguish three cases: The *high* income case, the *standard* case that defines the *range of income of interest*, and the *low* income case. In the high income case, $I \geq p'S^j$ and individual j maximizes his utility by buying his satiation vector, i.e., $X^j = S^j$. Otherwise, the budget constraint is tight and the procedure begins by forming a Lagrangian and setting its partials to zero,

$$\partial[U^j(X) - 2\lambda(p'X)]/\partial X = 0, \tag{1.3.0}$$

which is then solved to determine X^j as a function of λ; the expression for X^j as a vector function of λ is then substituted into (1.2.0) with the budget tight and solved for λ. If $X^j > 0$, then this is the standard case. The low case occurs, by definition, when the budget $I \leq I_j^*$ is so low that j maximizes his utility on the boundary of the non-negative orthant $X \geq 0$ by setting to zero one or more components of X.

We will loosely refer to the income levels between $I^* = \max I_j^*$ and $I^{**} = \min p'S^j$ as the *range of income of interest*. It is the range of income for consumption in which no individual j maximizes his utility on the boundary of the orthant $X \geq 0$ or has sufficient income to buy his satiation vector S^j. This range depends on the prices p. We think of this range as very broad, I^* representing extreme poverty and I^{**} as being very rich. In the context of the full model (Part IV), it is possible to have all individuals receive at least a minimum fixed consumption vector, so that $X^j(k) = 0$ should not be interpreted as j going without food, for example. Income, in this context, means income for purchases above this floor.

For the "standard" case, we substitute the quadratic expression for $U^j(X)$ given by (1.1.0) into (1.3.0) and differentiate partially:

$$M^j(S^j - X^j) = \lambda \cdot p. \tag{1.4.0}$$

Solving for $S^j - X^j$:

$$S^j - X^j = \lambda \cdot H^j p, \qquad \text{where } H^j = (M^j)^{-1}. \tag{1.5.0}$$

Note that the inverses $H^j = (M^j)^{-1}$ exist and are also *symmetric* and *positive definite*. We can now use the tight budget constraint to determine λ. Multiplying (1.5.0) by p' on the left and

setting $p'X^j = I$, we can solve for λ and substitute back into (1.5.0). This yields (1.6.0) below, the *demand function of individual* j as a function of prices and income I where λ, equal to the term in parentheses, is positive if the budget I is less than the income $p'S^j$ required to buy the satiation vector S^j. Note $p'H^jp > 0$ follows from H^j being positive definite. It is easy to see that (1.6.0) satisfies two properties of demand functions which hold when the budget constraint is tight: (a) $p'X^j = I$ and (b), X^j remains invariant if we rescale prices p and income I proportionally.

Theorem 1.1. *The demand function of individual j, for fixed prices p, is a piecewise linear function of consumption income I. (i) For a range that includes the "range of income of interest", the demand function is given by*

$$S^j - X^j = \left(\frac{p'S^j - I}{p'H^jp}\right)H^jp, \qquad I_j^* \le I \le I_j^{**} = pS^j \,. \tag{1.6.0}$$

(ii) For $I \le I_j^*$, one or more components $X^j(k) = 0$. As $I \to 0$, $X^j \to 0$. (iii) For $I > I_j^{**} = p'S^j$, the budget constraint is slack and $X^j = S^j$ for all $I \ge I_j^{**}$.

Proof. The theorem is a restatement of a well known result from quadratic programming. We therefore omit details and sketch only important steps in the proof. The Kuhn-Tucker conditions for testing whether $X = X(I)$ is optimum consists in partitioning the inequalities generated by taking partial derivatives into a set Ω that are tight under the current solution and $\bar{\Omega}$, those that are not. These are in $1-1$ correspondence with components $X_k > 0$ and $X_k = 0$. Call the corresponding index sets k also Ω and $\bar{\Omega}$. Suppose for two income levels $I = \alpha$ and $I = \beta$ the corresponding index sets Ω are the same for the two optimum solutions. Then the convex combination $X = \lambda X(\alpha) + \mu X(\beta)$, $I = \lambda\alpha + \mu\beta$, is optimal for all $\lambda + \mu = 1$, $(\lambda, \mu) \ge 0$. It follows that X is linear in I over segments where Ω sets are the same, there being at most one segment for each such Ω. For other Ω there may be only one I. The former are the broken line segments and the latter are the values of I corresponding to the break points. Since the number of different Ω is finite, there are a finite number of breakpoints and broken line segments. ∎

Expected Demand Function for Individuals at the Same Income Level. Our next step is to derive the functional form of the Engel curves, namely the average demand for some item k for all individuals with the *same* income level I as a function of I and prices p. For this purpose, as we already noted, we assume that the utility functions which people have are independent of the particular income which they happen to have. Suppose there are n_I individuals j at income level I. We need to have a way to assign the parameters (S^j, H^j) to their utility functions independent of their income I, next find $X^j = X$ which maximizes their utility subject to their budget constraint I, and then average X^j to compute the average demand vector as a function of I and fixed prices p.

Population Urn. We can achieve this independence of income by selecting in the parameter space (S, H) a representative set of n possible points (S^i, H^i) and writing them as labels on n balls, $i = (1, \ldots, n)$, and placing the balls in an imaginary urn which we call *the population urn*. In making a random drawing with replacement from the urn, we assume each (S^i, H^i) is equally likely to be drawn and assigned to individual j. If it is desired to make it more likely to choose certain (S^i, H^i), this could be done by weighing the distribution of (S^i, H^i) in the urn or by replication of certain of the balls. For convenience we have made this distribution discrete and n finite, but a general probability measure could be used instead.

Assumption 2, the Population Urn Assumption. *The utility function which individual j happens to have, is uncorrelated with I, his personal income for consumption. We achieve this by assigning to j a utility function with parameters $(S^j, H^j) = (S^i, H^i)$ where (S^i, H^i) is randomly drawn with replacement from an "urn"; we also assume the distribution in the urn of the "satiation-level" parameters S^i is independent of the distribution of the price "cross- effect" parameters $H^i = (M^i)^{-1}$.*

For any of the n_I individuals j, with income I, we denote their *expected* S^j and X^j by $\mathcal{E}S^j = \bar{S}^I$ and $\mathcal{E}X^j = \bar{X}^I$ respectively. Our assumption that the distribution of utility functions does not depend on I implies that this expected $\bar{S}^I$ is the same for all I and therefore $\bar{S}^I = \bar{S}$ where $\bar{S}$ is the arithmetic mean of the S^i in the population urn.

A special symbol $\mathcal{E}_i S^i$ is used to denote the *arithmetic mean* of S^i in the population urn. It has the same value as the expected value of a single random drawing of S^i from the population urn. Likewise, we denote the *arithmetic mean* of any function $\psi(S^i, H^i)$ for $i = 1, \ldots, n$ in the population urn by $\mathcal{E}_i\psi(S^i, H^i) = (1/n)\sum_1^n \psi(S^i, H^i)$. For the n_I individuals j that have an income I in the *range of income of interest*, $\max I_i^* = I^* \leq I \leq I^{**} = \min I_i^{**}$, we know that (1.6.0) holds no matter what (S^i, H^i) has been assigned to j from the urn. Therefore taking the expectation of (1.6.0) we obtain (1.7.1) below.

Theorem 1.2. *For fixed prices $p = p_t$ and for all individuals j whose income level I satisfies $\max I_i^* = I^* \leq I \leq I^{**} = \min p'S^i$ for (S^i, H^i) in the population urn, their expected consumption vector $\bar{X}^I$ is a linear function of I:*

$$\bar{S} - \bar{X}^I = \mathcal{E}_i(p'S^i)[\frac{1}{p'H^ip}H^i]p - I\mathcal{E}_i[\frac{1}{p'H^ip}H^i]p\,, \tag{1.7.1}$$

$$= (p\bar{S} - I)\mathcal{E}_i[\frac{1}{pH^ip}H^i]p\,, \qquad \bar{S} = \mathcal{E}_i S^i\,. \tag{1.7.2}$$

Proof. Our assumption that the parameters (S^j, H^j) assigned for individual j is chosen by a random drawing from the population urn independent of income I means, for fixed prices, that the expected values of the terms of (1.6.0) are those of (1.7.1); in particular the factor multiplying I in the second term does not depend on I. Therefore (1.7.1), at fixed prices, states that the Engel

curves which express each component k of X^I as a function of I are *linear* in I at fixed prices in the range $I^* \leq I \leq I^{**}$. Our additional assumption about the independence of the distributions of S^i and H^i in the urn implies (1.7.2) because

$$\mathcal{E}_i(p'S^i)[\frac{1}{p'H^ip}H^i]p = \mathcal{E}_i(p'S^i)[\mathcal{E}_i\frac{1}{p'H^ip}H^i]p = (p'\bar{S})[\mathcal{E}_i\frac{1}{p'H^ip}H^i]p \;. \quad \blacksquare \tag{1.7.3}$$

We denote the *difference* between the expected demand at incomes $p\bar{S}$ and $p\bar{S}-1$ by

$$G(p) = \mathcal{E}_i[\frac{1}{p'H^ip}H^i]p \tag{1.8.0}$$

It follows that the *difference* between the expected demand at saturation incomes $p'\bar{S}$ and I is $J \cdot G(p)$, where $J = pS - I$ is the *additional income* required to reach an income sufficient to buy the satiation vector. Substituting (1.8.0) into (1.7.2), a more complete statement of Theorem 1.2 is

Theorem 1.3. *For fixed prices $p = p_t$, then (i), for all individuals j whose income level I is in the range of income of interest, their expected consumption vector $\bar{X}^I$ is a linear function of I:*

$$\bar{S} - \bar{X} = (p'\bar{S} - I) \cdot G(p), \quad where \quad \max I_i^* = I^* \leq I \leq I^{**} = \min p'S^i \;. \tag{1.9.0}$$

(ii) As I decreases below I^*, it is more and more likely individuals j with this income will maximize their utility by setting components of X^j to zero and $\mathcal{E}X^j = \bar{X}^I \to 0$. (iii) As I increases beyond $I^{**} = \min p'S^i$, it is more and more likely individuals j will maximize their utility by buying their satiation vectors and $\mathcal{E}X^j = \bar{X}^I$ will level off to $\bar{S}$.

Comment. In Part V, we will present the empirical data of average consumption as a function of income level for eight broad consumer categories *Food, Clothing, Housing, Household Operation, Transportation, Recreation, Personal Care, and All Other.* We will see that the form of these Engel Curves is generally consistent with this theoretical result that the expected demand vector of individuals with same consumption income I is a linear function of I over the "range of income of practical interest," namely $\max I_i^* \leq I \leq \min p'S^i$.

Per capita demand as a function of per capita income. Assuming fixed prices in period t, the expected per capita demand function is derived from the expected demand of persons at various income levels by a convolution with the income distribution. Let $\bar{X}^I(k) = C_k(I)$ be the expected personal consumption per year of item k by individuals at the same income level I. Let $\phi(I)$ be the income distribution. The per capita consumption of item k and corresponding expected per capita income per year are given by

$$\bar{X}(k) = \int_{I=0}^{\infty} \phi(I) \cdot C_k(I)dI, \tag{1.10.1}$$

$$\bar{I} = \int_{I=0}^{\infty} \phi(I) \cdot I\, dI, \tag{1.10.2}$$

where $\bar{X}(k)$ denotes component k of $\bar{X}$. The symbol for expected *per capita* consumption vector $\bar{X}$ is to be distinguished from $\bar{X}^I$ which is the expected consumption vector of individuals j whose income level is I. It would appear that the correspondence between $\bar{X}$ and $\bar{I}$ depends on the distribution of ϕ; *in fact it does not* for a broad range of $\bar{I}$.

Theorem 1.4. *If $C_k(I)$ is a linear function of I, then independent of the distribution of income $\phi(I)$, $\bar{X}(k) = C_k(\bar{I})$, i.e., expected per capita consumption of item k is a linear function of per capita income $\bar{I}$ and this linear function is $C_k(\bar{I})$.*

Proof. Let $C_k(I) = a + bI$. Substituting into (1.10.1) and noting $\int \phi(I)dI = 1$, yields $\bar{X}(k) = a + b\bar{I} = C_k(\bar{I})$. ∎

Comment. The hypothesis that $C_k(I)$ is a linear function of I is only true by Theorems 1.2 and 1.3, for a restricted range of income which we have referred to as the "broad" *range of income of interest*, $\max I_i^* = I^* \le I \le I^{**} = \min pS^i$ for (S^i, H^i) in the population urn. Let us assume the distribution of income for consumption in the population is above the extreme poverty level $I^* = \max I_i^*$ and below being very rich, $I^{**} = \min pS^i$. As time goes by, per capita income $\bar{I}$ will change (and likely increase) and the distribution of income $\phi(I)$ about $\bar{I}$ will change. As long as people at the same income level at the same prices buy in the same way in the future and the income distribution change is not so drastic that some j have income I below their I_j^* or above their pS^j, Theorem 1.4 states that $\bar{X}(k)$ is the *same* linear function of $\bar{I}$ as $X^I(k)$ is of I for some range $\bar{I}^* \le \bar{I} \le \bar{I}^{**}$. Therefore, we have established:

Theorem 1.5. *For fixed prices $p = p_t$, (i) expected per capita consumption $\bar{X}$ is a linear function of per capita income $\bar{I}$ for a certain range of income $\bar{I}$; namely:*

$$\bar{S} - \bar{X} = (p'\bar{S} - \bar{I}) \cdot G(p) , \qquad \bar{I}^* \le \bar{I} \le \bar{I}^{**} , \tag{1.11.1}$$

where $G(p) = H(p) \cdot p$ and $H(p) = \mathcal{E}_i(p'H^ip)^{-1} \cdot H^i$ is a symmetric positive-definite matrix whose elements depend on p and not on $\bar{I}$.

(ii) *If $\bar{I} < \bar{I}^*$, then it is more and more likely that some individual's income $I < I_j^*$, and these j will maximize their utility by setting some components of X^j to zero; as $\bar{I} \to 0$, $\bar{X} \to 0$.* (iii) *If $\bar{I} > \bar{I}^{**}$, then it is more and more likely that some individual's income $I > p'S^j$; these j will maximize their utility by buying their satiation vectors S^j; for sufficiently high $\bar{I}$, $\bar{X}$ will level off to $\bar{S}$.*

Note especially that our predicted result of linearity *does not depend on the shape of the income distribution or how this changes as per capita income $\bar{I}$ increases in the future* providing it does not rise so high that some j have income I such tht $p'S^j < I$ or decreases so low that some $I_j^* > I$. Implicit, of course, is the assumption that tastes don't change with time. If they do, there is no problem adjusting the model for trends in taste.

The empirical studies of Avriel and McAllister [2], based on special assumptions about how income distribution will change relative to $\bar{I}$ in the future are consistent with the results we have just derived without their special assumptions.

PART II:
NECESSARY AND SUFFICIENT CONDITION FOR INTEGRALITY OF THE EXPECTED PER-CAPITA DEMAND FUNCTION

Our immediate objective is to derive a necessary and sufficient condition that an *inverse* expected per capita demand function and *utility* function exist for period t when the expected demand function given by (1.11.1). The condition for existence is closely related to one given by Slutsky, see Varian [19]. Our proof is specific to the demand function given by (1.11.1). It is therefore assumed in the discussion that follows that per capita income is in the "range of income of interest" $\bar{I}^* \le \bar{I} \le \bar{I}^{**}$.

Definition of a Utility Function. In order for $\bar{U}(\bar{X})$ to qualify as a per capita utility function, we let $v = \bar{S} - \bar{X}$ and require $Z(v) = -\bar{U}(\bar{X})$ to be a *convex function* twice differentiable (except possibly at $v = 0$) which attains a minimum subject to a budget constraint $p'\bar{X} = \bar{I}$, or equivalently $p'v = J$, where $J = p'\bar{S} - \bar{I}$, at a unique finite point $v = v^*$. Therefore v^* is a function of p and J which we denote by $\bar{S} - \bar{X} = v(p, J)$ and call the latter the *demand function* associated with the utility function $\bar{U}(\bar{X})$.

Conversely, if we are given a demand function $\bar{S} - \bar{X} = v(p, J)$, such as

$$\bar{S} - \bar{X} = v = J \cdot \underset{i}{\mathcal{E}}\,(p'H^i p)^{-1} \cdot H^i p \quad \text{where} \quad J = p'\bar{S} - \bar{I}\,, \tag{2.1.0}$$

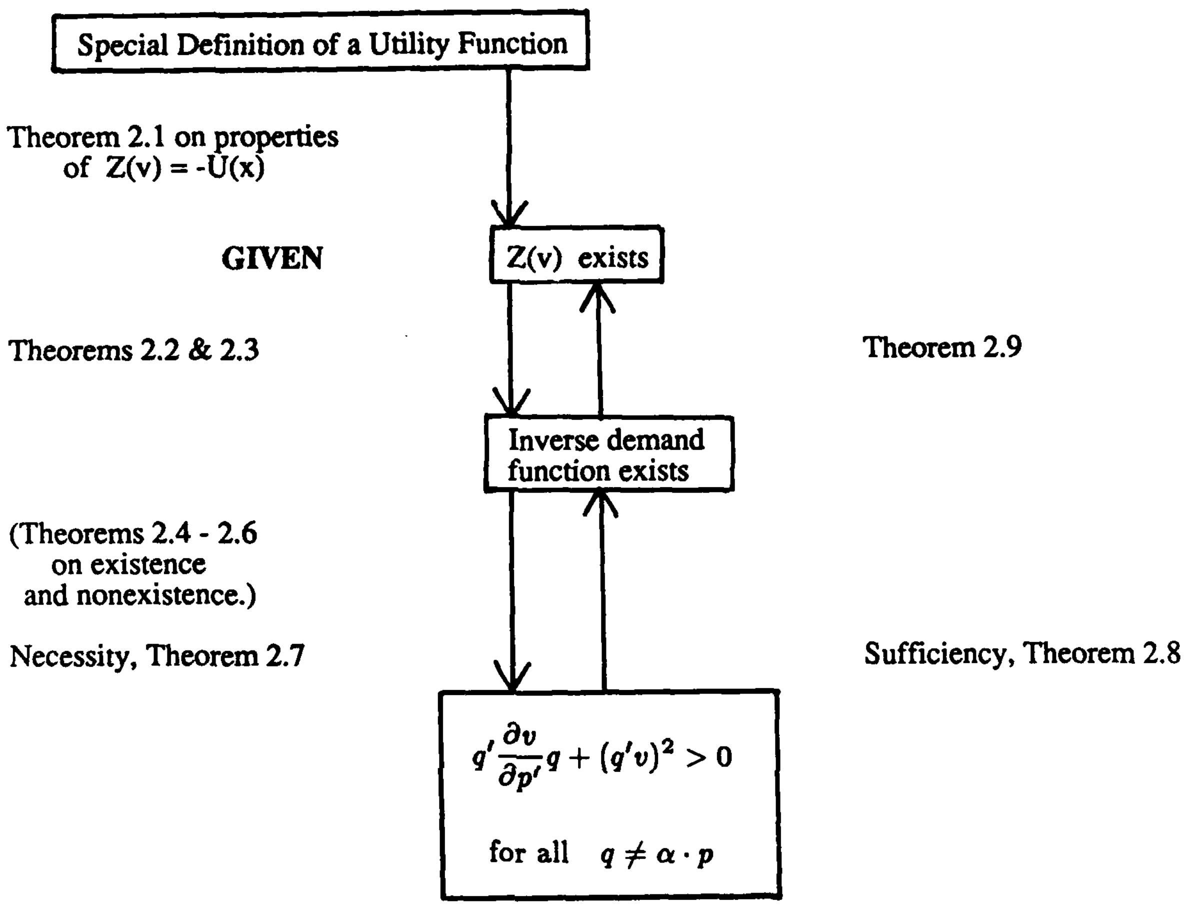
we inquire if there exists a utility function with which the demand function is associated. If yes, we say that the demand function is *integrable*.

We will, however, *restrict* the utility functions considered from now on to those *homogeneous in* v *of degree* $+1$. To conform with this definition, we redefine the utility function (1.1.2) for an individual to be $U^j(X) = -[(S^j - X)'M^j(S^j - X)]^{1/2}$. We call a function a **per capita utility function**, $\bar{U}(\bar{X}) = -Z(v)$, $v = \bar{S} - \bar{X}$, if (i) $Z(v)$ is twice differentiable for all v (except possibly at $v = 0$); (ii) $Z(v)$ is a homogeneous function in v of degree 1 along every ray, i.e., $Z(\alpha v) = \alpha Z(v)$ for all v, $\alpha \ge 0$; (iii) $Z(v)$ is strictly convex between any two points $v^1 \ne v^2$ satisfying the budget constraint $p'v = 1$:

$$\lambda Z(v^1) + \mu Z(v^2) > Z(\lambda v^1 + \mu v^2) \quad \text{for all} \ (\lambda > 0,\ \mu > 0,\ \lambda + \mu = 1)\,; \tag{2.2.0}$$

and (iv) $Z(v) > 0$ for all $v \ne 0$.

We first show that $Z(v)$ under the second definition qualifies as a utility function under the first definition. We then seek a necessary and sufficient condition that $\bar{U}(\bar{X})$ exists when the demand function is given by (2.1.0). The diagram on the next page outlines the logical dependence of the various theorems upon one another which yield or are implied by this condition.



Theorem 2.1. *$\min Z(v)$ subject to $p'v = 1$ is attained at a finite point $v = v^*$ and is unique. Denoting $v(p,1) = v^*$, the expected per capita demand function associated with $Z(v)$ is $\bar{S} - \bar{X} = v(p,J) = J \cdot v(p,1)$ where $J = (p\bar{S} - \bar{I})$.*

Proof. Assume on the contrary that $\inf Z(v)$ is not attained at any finite point. In this case, there exists a strictly decreasing sequence

$$Z(v^1) > Z(v^2) > \cdots > Z(v^t) > \cdots > \inf Z(v) \tag{2.3.0}$$

such that $Z(v^t) \to \inf Z(v) \geq 0$ as $\|v^t\| \to +\infty$, and such that $p'v^t = 1$ for all t. A subsequence can be chosen, so that normalized vectors $v^t/\|v^t\| \to v^0$, where $v^0 \neq 0$ and $Z(v^0) > 0$. For this subsequence, due to homogeneity of $Z(v)$,

$$Z(v^t) = \|v^t\| \cdot Z(v^t/\|v^t\|) \to \|v^t\| \cdot Z(v^0) \to +\infty \ , \tag{2.3.1}$$

contradicting that $Z(v^t)$ is a strictly decreasing subsequence.

Therefore $Z(v)$ attains its minimum at *either* a unique finite point v^* *or* attains it at least two finite points $v^1 \neq v^2$. Now v^1 and v^2 cannot be on the same ray and both satisfy the budget constraint $p'v = 1$. If not on the same ray, then by (2.2.0), any convex combination of $\lambda v^1 + \mu v^2$ satisfies the budget constraint and yields a lower $Z(v)$, contradicting $Z(v^1) = Z(v^2)$ are minimum values.

To prove $v(p, J) = J \cdot v(p, 1)$, following the same proof as above, we know that min $Z(v)$ subject to a budget constraint $p'v = J$ or $(p'/J)v = 1$ is attained at a unique point, $v^4 = v(p, J)$. Let $v^1 = v(p, 1)$ and assume on the contrary:

$$Z(J \cdot v^1) = JZ(v^1) > Z(v^4) = JZ(v^4/J) , \tag{2.3.2}$$

implying that v^4/J, which satisfies $pv = 1$, maps into $Z(v^4/J) < Z(v^1)$ a contradiction. ∎

We defined p_t as *fixed* period t prices. Let p be a rescaled price vector that maps into v by the relation: $v = G(p)$. We have the following definitions and relations:

$$\bar{S} - \bar{X} = v = G(p) , \tag{2.4.0}$$

$$G(p) = \mathcal{E}_i(p'H^ip)^{-1} \cdot H^ip \tag{2.4.1}$$

$$H(p) = \mathcal{E}_i(p'H^ip)^{-1} \cdot H^i \tag{2.4.2}$$

$$p'v \equiv 1 \qquad \text{for all } p \neq 0 . \tag{2.4.3}$$

Two lemmas about $G(p)$ and $H(p)$:

Lemma 2.1. *Elements (k, ℓ) of the matrix $H(p) = \mathcal{E}_i(p'H^ip)^{-1} \cdot H^i$ are homogeneous functions of p of degree -2; $H(p)$ is square, symmetric, positive definite, moreover $p'[H(p)]p = p'G(p) = p'v \equiv 1$ for all $p \neq 0$.*

Proof. By definition, element (k, ℓ) of $H(p)$ is $\mathcal{E}_i(p'H^ip)^{-1}H^i_{k\ell}$ and is equal to element (ℓ, k) since $H^i_{k\ell} = H^i_{\ell k}$. It is positive definite because the assumed positive definiteness of H^i implies $q'[H(p)]q = \mathcal{E}_i(p'H^ip)^{-1}(q'H^iq) > 0$ for all $p \neq 0$, $q \neq 0$. In particular $p'[H(p)]p = \mathcal{E}_i(p'H^ip)^{-1}(p'H^ip) = 1$. ∎

Lemma 2.2. *The elements of the vector $v = G(p) = H(p)p = \mathcal{E}_i(p'H^ip)^{-1} \cdot H^ip$ are homogeneous functions of degree -1. The matrix $\partial v/\partial p'$ is square and symmetric.*

Proof. We define element (k, ℓ) of $\partial v/\partial p'$ as $\partial v_k/\partial p_\ell$ where v_k is k-th component of v and p_ℓ the ℓ-th component of p. From (2.4.0), (2.4.1):

$$v_k = \mathcal{E}_i(p'H^ip)^{-1} \cdot H^i_{k\cdot}p , \tag{2.5.0}$$

where $H^i_{k\cdot}$ denotes the k-th row of H^i. Clearly v_k is a homogeneous function in p of degree -1. Taking partials,

$$\frac{\partial v_k}{\partial p_\ell} = \mathcal{E}_i\left[\frac{H^i_{k\ell}}{p'H^ip} - \frac{(H^i_{k\cdot}p)(p'H^i_{\cdot\ell} + H^i_{\cdot\ell}p)}{(p'H^ip)^2}\right] = \frac{\partial v_\ell}{\partial p_k} \tag{2.5.1}$$

where the ℓ-th column $H^i_{\cdot\ell} = (H^i_{\ell\cdot})'$ and $H^i_{k\ell} = H^i_{\ell k}$ because H^i is symmetric. ∎

Theorem 2.2. *The unique v that minimizes $Z(v)$ subject to the budget constraint $p'v = 1$ satisfies the first-order condition:*

$$\frac{1}{Z} \cdot \frac{\partial Z}{\partial v} = p \,. \tag{2.6.0}$$

Conversely, given any $v = v^*$, there exists a unique p such that min Z *subject to $p'v = 1$ is attained at v^*, namely:*

$$\frac{1}{Z} \cdot \frac{\partial Z}{\partial v}\Big|_{v=v^*} = p \,. \tag{2.6.1}$$

Proof. . Forming the Lagrangian $Z(v) - \lambda(p'v)$ and setting its partial derivates to zero, we obtain $\partial Z/\partial v = \lambda \cdot p$ where λ is chosen so that $p'v = 1$. Applying Euler's Theorem for homogeneous functions to $Z(v)$ of degree $+1$:

$$Z = v'(\frac{\partial Z}{\partial v}) = v'(\lambda p) = \lambda(v'p) = \lambda \,, \tag{2.6.2}$$

whence (2.6.0). Conversely, given $v = v^*$, the p defined by (2.6.1) satisfies $v'p = v'(\partial Z/\partial v)/Z = Z/Z = 1$ by Euler's Theorem. Suppose now for p given by (2.6.1) that min $Z(v)$ subject to $p'v = 1$ is attained at some other $v = v^0 \neq v^*$ satisfying (2.6.0). Then since $Z(v)$ is convex and differentiable, both v^0 and v^* satisfy necessary and sufficient conditions (2.6.0) to be global minima points, contradicting uniqueness established in Theorem 2.1. ∎

Theorem 2.3. *A necessary condition for the existence of a utility function $\bar{U}(\bar{X}) = -Z(v)$, associated with the demand function*

$$\bar{S} - \bar{X} = J \cdot G(p) \,, \quad J = p\bar{S} - \bar{I} \,, \tag{2.7.0}$$

is that the inverse function of $v = G(p)$, exists, namely,

$$p = G^{-1}(v) = \frac{1}{Z}\frac{\partial Z}{\partial v} \,, \quad p'v = 1 \,, \quad v = \bar{S} - \bar{X} \,. \tag{2.7.1}$$

Proof. If $v = G(p)$, then by (2.4.3) $p'v \equiv 1$ for all p. If a utility function $\bar{U}(\bar{X}) = -Z(v)$ exists associated with the demand function (2.7.0), then minimizing $Z(v)$ subject to $p'v = 1$ satisfies (2.6.0). Since Z is a function of v, (2.7.1) states that p can be expressed as a function of v when $p'v = 1$. Hence a necessary condition for the existence of a utility function is that $v = G(p)$ have an inverse function $p = G^{-1}(v)$. ∎

Comment. There are two ways that the function $v = G(p)$ can fail to have an inverse. The first is: given v, there exists no p that satisfies the equation $v = G(p)$. The second way is: given a particular v, there is more than one p satisfying the equation. We will prove that first way can

never happen and the second way can. However, under certain conditions that we will specify later, there is a unique solution for all choices of $v \neq 0$.

Theorem 2.4. *Given any $v \neq 0$, there are always one or more $p \neq 0$ satisfying*

$$v = G(p) = \mathcal{E}_i(p' H^i p)^{-1} \cdot H^i p\,, \quad v \neq 0\,. \tag{2.8.0}$$

Proof. Let $i = (1, \ldots, n)$ and let $\Omega = \{\lambda_i : \lambda_i \geq 0,\ \sum_1^n \lambda_i = 1\}$ be the $n-1$ dimensional simplex. We now define a continuous mapping $\lambda \to \mu \in \Omega$: Choose any $\lambda \in \Omega$; let $\tilde{H} = \sum_1^n \lambda_i H^i$; determine $\hat{p} = n\tilde{H}^{-1}v$. Note that $\tilde{H}$ is positive definite because it is a convex combination of positive definite matrices H^i; hence $\tilde{H}^{-1}$ exists; hence $\hat{p}$ can be computed. Next, form $\bar{\mu}_i = (\hat{p}' H^i \hat{p})^{-1} > 0$ for each i and set $\mu_i = \bar{\mu}_i / \sum_1^n \bar{\mu}_k$. It is not difficult to show that the mapping $\lambda \to \mu$ is continuous. According to Brouwer's Theorem, a *fixed point* $\lambda^0 \to \lambda^0$ exists. For this $\lambda^0 = \mu$ determine

$$\hat{p} = n\tilde{H}^{-1}v \qquad \text{where } \tilde{H} = \Sigma_1^n \lambda_i^0 H^i\,, \tag{2.8.1}$$

$$\mu_i = \lambda_i^0 = (\hat{p}' H^i \hat{p})^{-1} / \Sigma_1^n (\hat{p}' H^i \hat{p})^{-1}\,, \tag{2.8.2}$$

$$\begin{aligned} nv &= \tilde{H}\hat{p} = \Sigma_1^n \lambda_i^0 H^i \hat{p} \\ &= [\Sigma_1^n (\hat{p}' H^i \hat{p})^{-1} \cdot H^i \hat{p}] / \Sigma_1^n (\hat{p}' H^i \hat{p})^{-1} \end{aligned} \tag{2.8.3}$$

$$v = (1/n)\Sigma_1^n (p' H^i p)^{-1} \cdot H^i p = G(p)\,. \tag{2.8.4}$$

where $p = \hat{p} \cdot \Sigma_1^n (\hat{p}' H^i \hat{p})^{-1}$, whence (2.8.0). ∎

Theorem 2.5. *Given certain v and certain H^i, it is possible there exist more than one p satisfying $v = G(p)$, implying $G^{-1}(v)$ does not exist in general.*

Proof. Let $n = 2$ where $i = 1, \ldots, n$ and let $m = 2$ where matrices H^i are $m \times m$. Let

$$H^1 = \begin{pmatrix} 2.0 & -.6 \\ -.6 & 0.2 \end{pmatrix}, \quad H^2 = \begin{pmatrix} 0.2 & -.6 \\ -.6 & 2.0 \end{pmatrix}, \quad v = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \tag{2.9.0}$$

In the case of $n = 2$ and $m = 2$, it is not difficult to show that there are three different p satisfying $v = (1/2)\sum_1^2 (p' H^i p)^{-1} \cdot H^i p$ generated by three different choices of λ_1^0 and corresponding $\lambda_2^0 = 1 - \lambda_1^0$, see (2.8.1) and (2.8.2). One of these three values of λ_1^0 can be real and the other two conjugate pairs, $\alpha \pm \beta\sqrt{-1}$, or all three can be real. For the special case (2.9.0), there are three real solutions:

$$p = (1, 1)\,, \quad p \doteq (.48382,\ 1.51618)\,, \quad p \doteq (1.51618,\ .48382)\,.$$

Since these solutions satisfy $v = G(p)$, all three satisfy $p'v \equiv 1$. ∎

Theorem 2.6. *For some choices of H^i, $v = G(p)$ has an inverse $p = G^{-1}(v)$.*

Proof. Assume $H^i = \bar{H}$ for all i. It is easy to verify that a p which satisfies (2.10.0) is given by (2.10.1):

$$v = (p'\bar{H}p)^{-1}\bar{H}p\,. \tag{2.10.0}$$

$$p = (v'\bar{H}^{-1}v)^{-1}\cdot\bar{H}^{-1}v\,, \tag{2.10.1}$$

The RHS of (2.10.1) is $G^{-1}(v)$ because, as we will now prove, this solution is unique: Assume, on the contrary, $\tilde{p} \neq p$ also satisfies (2.10.0):

$$v = (\tilde{p}'\bar{H}\tilde{p})^{-1}\bar{H}\tilde{p}\,. \tag{2.10.2}$$

Substituting this expression for v into (2.10.1), we obtain $p = \tilde{p}$, a contradiction. ∎

When $v = G(p)$ has an inverse $p = G^{-1}(v)$, we will make use of the following lemmas.

Lemma 2.3. *$v = G(p)$ and $p = G^{-1}(v)$ are homogeneous functions of degrees $\rho = -1$ and $1/\rho = -1$ in p and v respectively.*

Lemma 2.4. *If the matrix $\partial v/\partial p'$ is non-singular at p, then its inverse $\partial p/\partial v'$ exists.*

Proof. $\partial v/\partial v = \text{identity} = [\partial v/\partial p']\cdot[\partial p/\partial v']$ from the theory of implicit functions. ∎

Lemma 2.5. *The matrix $\partial p/\partial v'$ is symmetric.*

Proof. This follows from Lemmas 2.2 and 2.4. ∎

Lemma 2.6.

$$v = -[\partial v/\partial p']p\,; \quad v' = -p'[\partial v/\partial p']\,; \tag{2.11.0}$$

$$p = -[\partial p/\partial v']v\,; \quad p' = -v'[\partial p/\partial v']\,. \tag{2.11.1}$$

Proof. The first part of the Lemma follows from Euler's Theorem for homogeneous forms of degree -1, see Lemma 2.3, and the second part follows from the first part and symmetry, see Lemma 2.5. ∎

Lemma 2.7.

$$p'\frac{\partial v}{\partial p'}p + (p'v)^2 \equiv 0 \tag{2.12.0}$$

Proof. By (2.11.0), the first term is $-p'v$ and $p'v = 1$ by (2.4.3). ∎

Theorem 2.7. *Given $v = G(p)$, a necessary condition that a disutility function $Z(v)$ exists is*

$$q' \frac{\partial v}{\partial p'} q + (q'v)^2 > 0 \quad \textit{for all } q \neq \alpha p, \ p \neq 0 \,. \tag{2.12.1}$$

Proof. By definition of a utility function, its correpsonding disutility function satisfies $Z(v) > 0$ for all $v \neq 0$ and $\partial^2 Z/\partial v^2$ is a positive semi-definite matrix such that:

$$v' \frac{\partial^2 Z}{\partial v^2} v = 0 \,, \quad u' \frac{\partial^2 Z}{\partial v^2} u > 0 \,, \quad \text{for all } u \neq \alpha v \,. \tag{2.12.2}$$

We have shown by (2.7.1):

$$p = (\partial Z/\partial v)/Z \,. \tag{2.12.3}$$

At any $v \neq 0$, $\partial p/\partial v'$ exists since $\partial p/\partial v' = (1/Z)\partial^2 Z/\partial v^2 - (1/Z)^2(\partial Z/\partial v)^2$ and $Z(v) \neq 0$. From $\partial^2 Z/\partial v^2 = \partial[\partial Z/\partial v]/\partial v'$ and (2.12.3):

$$\frac{\partial^2 Z}{\partial v^2} = \frac{\partial p}{\partial v'} Z + p \frac{\partial Z}{\partial v'} = \frac{\partial p}{\partial v'} \cdot Z + (pp') \cdot Z \tag{2.12.4}$$

$$= Z \cdot [\frac{\partial p}{\partial v'} + pp'] \tag{2.12.5}$$

By (2.5.1), the inverse of $\partial p/\partial v'$, namely $\partial v/\partial p'$, also exists, at any point $v \neq 0$, $p \neq 0$. Therefore, for $u \neq \alpha v$, we can find a q satisfying

$$u = \frac{\partial v}{\partial p'} q \,, \quad q \neq \alpha \cdot p \,. \tag{2.12.6}$$

$Z(v)$, by assumption, is strictly convex except along rays. We therefore require

$$0 < u' \frac{\partial^2 Z}{\partial v^2} u = Z\big[u' \frac{\partial p}{\partial v'} u + (u'p)^2\big] \quad \text{for all } u \neq \alpha v, \quad v \neq 0 \,, \tag{2.12.7}$$

$$= Z\big[q' \frac{\partial v}{\partial p'} q + (q'v)^2\big] \quad \text{for all } q \neq \alpha \cdot p, \quad p \neq 0 \,, \tag{2.12.8}$$

by (2.12.6). Whence (2.12.1) since $Z > 0$. ∎

Theorem 2.8. *A sufficient condition that the inverse function $p = G^{-1}(v)$ exists when $v = G(p) = \mathcal{E}_i (pH^i p)^{-1} \cdot H^i p$ is*

$$q' \frac{\partial v}{\partial p'} q + (q'v)^2 > 0 \quad \textit{for all } q \,, \ q \neq \alpha \cdot p \,, \ p \neq 0 \,. \tag{2.13.0}$$

Proof. We first show that the function $W(p)$, defined by

$$\log W(p) = \tfrac{1}{2} \mathcal{E}_i \log(p' H^i p) \,, \quad p \neq 0 \,, \tag{2.13.1}$$

under the hypothesis (2.13.0), is (a) convex; (b) $W(p) > 0$ for $p \neq 0$; (c) its Hessian $\partial^2 W/\partial p^2$, is strictly positive definite for all directions $q \neq \alpha \cdot p$; and finally, (d) $p'(\partial^2 W/\partial p^2)p = 0$. Clearly $W(p) > 0$. The first partial is:

$$\frac{1}{W} \cdot \frac{\partial W}{\partial p} = \mathcal{E}_i(p'H^ip)^{-1} \cdot H^ip \,, \tag{2.13.2}$$

$$\frac{\partial W}{\partial p} = W \cdot G(p) \,. \tag{2.13.3}$$

Letting $v = G(p)$, $\partial W/\partial p = Wv$. Applying (2.13.3), the second partial is:

$$\frac{\partial^2 W}{\partial p^2} = W[(\frac{\partial v}{\partial p'}) + v \cdot v'] \,, \tag{2.13.4}$$

$$q'[\frac{\partial^2 W}{\partial p^2}]q = W[q'(\frac{\partial v}{\partial p'})q + (q'v)^2] > 0 \,, \tag{2.13.5}$$

by hypothesis (2.13.0) and $W > 0$. Finally, if we replace q by p in (2.13.5), then by Euler's Theorem, $p'(\partial v/\partial p')p = p'(-v) = -1$ and $(p'v)^2 = +1$, whence

$$p'\frac{\partial^2 W}{\partial p^2}p = 0 \,. \tag{2.13.6}$$

Conditions (2.13.5) and (2.13.6), and $W > 0$ imply that $\partial^2 W/\partial p^2$ given by (2.13.4) is positive semi-definite.

We now use the properties of $W(p)$ to show for any given v there exists a *unique* p such that $v = G(p)$. Suppose not true and for some given $v \neq 0$, there exists two vectors $p = \bar{p}$ and $p = \hat{p}$ such that $v = G(p)$ is satisfied. Consider now the problem of

$$\min W(p) \qquad \text{subject to } \; v'p = 1 \,, \tag{2.14.0}$$

where v is given and p is variable. Since $W(p)$ is convex, we find its minimum at some $p = p^*$ by setting the partials of the Lagrangian $W(p) - \lambda(v'p)$ to zero obtaining

$$\frac{\partial W}{\partial p} = \lambda v = Wv \,, \tag{2.14.1}$$

where $\lambda = W$ because $W = p'(\partial W/\partial p) = p'(\lambda v) = \lambda(p'v) = \lambda$. Equation (2.13.3) and (2.13.1) together expresses $\partial W/\partial p$ in terms of p while (2.14.1) places a condition on p to qualify as a minimizing p^*. Equating (2.14.1) with (2.13.3), we see that p^* satisfies $v = \mathcal{E}_i(p'H^ip)^{-1} \cdot H^ip$, and therefore $p^* = \bar{p}$ and $p^* = \hat{p}$ are both optimal.

But minimizing of a convex function $W(p)$ subject to linear constraint $v'p = 1$ can only have global minima. Therefore, $W(\bar{p})$ and $W(\hat{p})$ are both global minima and $W(\bar{p}) = W(\hat{p})$. Vectors $\bar{p}$ and $\hat{p}$ cannot lie on same ray and both satisfy the budget constraint (2.14.0). Therefore, by (2.13.5), in the direction $q = \bar{p} - \hat{p}$ from $\bar{p}$ to $\hat{p}$, $(\bar{p} - \hat{p})'[\partial^2 W/\partial p](\bar{p} - \hat{p}) > 0$ for all $p = \lambda\bar{p} + \mu\hat{p}$, $\lambda + \mu = 1$, $(\lambda, \mu) > 0$. Because of strict convexity along this segment, $W(p) < W(\bar{p}) = W(\hat{p})$ holds; also $\bar{p}'v = p'v = \hat{p}'v = 1$ holds, contradicting $\bar{p}$ and $\hat{p}$ both being minimizing points. ∎

Theorem 2.9. *A necessary and sufficient condition that there exists a utility function $\bar{U}(\bar{X}) = -Z(v)$, $v = \bar{S} - \bar{X}$, associated with the expected per capita demand function $\bar{S} - \bar{X} = (p'\bar{S} - \bar{I})\mathcal{E}_i(p'H^ip)^{-1} \cdot H^ip$ for period t is*

$$q' \frac{\partial v}{\partial p'} q + (q'v)^2 > 0 \,, \quad \text{for all } q \neq \alpha \cdot p \,, \ p \neq 0 \,. \tag{2.15.0}$$

The function $Z(v)$ is the mapping $v \to Z$ defined by the following procedure:

Step 1. *Find the unique p satisfying*

$$v = \mathcal{E}_i(p'H^ip)^{-1} \cdot H^ip \tag{2.15.1}$$

Step 2. *For any fixed scalar $\gamma > 0$, find Z satisfying*

$$\log \gamma Z = -\tfrac{1}{2}\mathcal{E}_i \log(p'H^ip), \quad \gamma > 0 \,, \tag{2.15.2}$$

Proof. Necessity of (2.15.0) has already been demonstrated. Therefore we assume (2.15.0) is true; by Theorem 2.8, $p = G^{-1}(v)$ exists, hence p of (2.15.1) is unique. Our objective is to show that $\hat{Z}(v) = \gamma Z(v)$, $\gamma > 0$, is a disutility function associated with $v = G(p)$. To qualify by our definition as a disutility function, we must show properties A, B, C, D, E below:

A. $\hat{Z}(v) > 0$ for all $v \neq 0$. This follows by rewriting (2.15.2),

$$\hat{Z} = [(p'H^1p) \cdot (p'H^2p) \cdots (p'H^np)]^{-1/2n} > 0 \,, \quad \text{for all } p \neq 0 \,. \tag{2.15.3}$$

B. $\hat{Z}(v)$ is homogeneous in v of degree $= 1$. Proof: Note v is homogeneous in p of degree -1 by (2.15.1) and $\hat{Z}$ is homogeneous in p of degree -1 by (2.15.3), implying $\hat{Z}$ is homogeneous in v of degree 1.

C. Minimizing $\hat{Z}(v)$ subject to the budget $p'v = 1$ implies the demand function $v = G(p)$. Proof: According to Theorem 2.3, the inverse demand function associated with minimizing $\hat{Z}(v)$ subject to $p'v = 1$ is

$$\frac{1}{\hat{Z}} \frac{\partial \hat{Z}}{\partial v} = p \tag{2.15.4}$$

where the left hand side is viewed as a function of v. We must therefore verify in the case when $v = G(p)$ and $\hat{Z}$ is defined by (2.15.1) and (2.15.2) that the left hand side of (2.15.4) is indeed $G^{-1}(v)$. From (2.15.2):

$$\frac{1}{\hat{Z}} \frac{\partial \hat{Z}}{\partial v'} = \frac{\partial \log \hat{Z}}{\partial v'} = \frac{(-1/2)\partial[\mathcal{E}_i \log(p'H^ip)]}{\partial p'} \cdot [\frac{\partial p}{\partial v'}] \tag{2.15.5}$$

$$= -[\mathcal{E}_i(p'H^ip)^{-1} \cdot H^ip]' \cdot [\frac{\partial p}{\partial v'}]$$

$$= -v' \cdot [\frac{\partial p}{\partial v'}] = p' \tag{2.15.6}$$

where $\partial p/\partial v'$ refers to the functional relation between p and v implied by $v = G(p)$ and where the last step follows from Euler's Theorem for homogeneous forms, see (2.11.0). Therefore, the p generated by (2.15.4) is the same one that is related to v by $v = G(p)$.

D. The function $\hat{Z}(v)$ is convex. Proof: Following the steps of Theorem 2.7,

$$u'\frac{\partial^2 \hat{Z}}{\partial v^2}u = \hat{Z}[q'\frac{\partial v}{\partial p'}q + (q'u)^2] > 0\,, \quad u = \frac{\partial v}{\partial p'}q\,, \quad q \neq \alpha p\,, \quad p \neq 0\,. \tag{2.15.7}$$

According to our hypothesis (2.15.0) and $\hat{Z} > 0$ by (2.15.3), the Hessian $[\partial^2\hat{Z}/\partial v^2]$ is strictly positive definite in the direction of any two points not on the same ray. Note (2.12.0) holds along a ray.

E. General $Z(v) = \gamma\hat{Z}(v)$ where $\gamma > 0$ is a scalar constant. Proof: By Theorem 2.3 and (2.15.4), both general $Z(v)$ and $\hat{Z}(v)$ satisfy

$$\frac{1}{Z}\frac{\partial Z}{\partial v} = \frac{1}{\hat{Z}}\frac{\partial \hat{Z}}{\partial v} = p \tag{2.15.8}$$

Integrating

$$\log Z = \log\hat{Z} + \log\gamma\,, \quad \gamma > 0 \tag{2.15.9}$$

whence $Z = \gamma\cdot\hat{Z}$. From now on we will assume $\gamma = 1$ and $Z = \hat{Z}$. ∎

Theorem 2.10. *A necessary and sufficient condition that there exists a utility function $\bar{U}(\bar{X}) = -Z(v)$, $v = \bar{S} - \bar{X}$, associated with the expected per capita demand function $\bar{S} - \bar{X} = (p'\bar{S} - \bar{I})\cdot \mathcal{E}_i(p'H^ip)^{-1}\cdot H^ip$ for period t is*

$$\mathcal{E}_i[\frac{q'H^iq}{p'H^ip} - 2\frac{(q'H^ip)^2}{(p'H^ip)^2}] + [\mathcal{E}_i\frac{q'H^ip}{p'H^ip}]^2 > 0\,. \tag{2.16.0}$$

for all $p \neq 0$, $q \neq \alpha p$.

Proof. Substituting into (2.15.0), $v = \mathcal{E}_i(p'H^ip)^{-1}\cdot H^ip$. Noting (2.5.1),

$$\frac{\partial v}{\partial p'} = \mathcal{E}_i[\frac{H^i}{p'H^ip} - 2\frac{H^ip\cdot p'H^i}{(p'H^ip)^2}] \tag{2.16.1}$$

form which we obtain (2.16.0). ∎

PART III:
TWO SUFFICIENT CONDITIONS FOR INTEGRABILITY

Having found a necessary and sufficient condition that a per capita utility function exists, namely

$$\mathop{\mathcal{E}}_{i}\left[\frac{q'H^iq}{p'H^ip} - 2\frac{(q'H^ip)^2}{(p'H^ip)^2}\right] + \left[\mathop{\mathcal{E}}_{i}\frac{q'H^ip}{p'H^ip}\right]^2 > 0 \tag{3.1.0}$$

holds for all $p \neq 0$, $q \neq \alpha \cdot p$, we now seek conditions on H^i in the population urn that guarantee this. We prove two important theorems which show that the distribution of H^i in the population urn would have to be highly polarized as in the example of Theorem 2.5 in order for condition (3.1.0) to fail. It is not difficult to show if $H_i = \bar{H}$ for all i that (3.1.0) holds. Let $\bar{H}$ be any positive definite matrix, for example $\bar{H} = \mathcal{E}_i H^i$ where $\Sigma\Sigma H^i_{k\ell} = 1$. One measure of how much H^i differs from $\bar{H}$ is to form $\bar{H}^i = \bar{H}^{-1/2}H^i\bar{H}^{-1/2}$ and compare $\bar{H}^i$ to I, the identity matrix. The eigenvalues of I are all unity and the ratio of its hightest to lowest eigenvalues is unity. Therefore, we can study how much the ratio of the highest to lowest eigenvalues of $\bar{H}^i$ has to differ from unity before condition (3.1.0) is violated.

Indeed we will show in Theorem 3.4 that this ratio would have to be greater than $3 + \sqrt{8}$ for each i in order for condition (3.1.0) to fail. But even this will not cause failure when the axes of various ellipsoids $pH^ip = constant$ are randomly rotated to some extent with respect to each *other; if the rotations are uniformly* distributed (or nearly so) in R^n, then we will show in Theorem 3.5 that regardless of what the ratio of highest to lowest eigenvalues of $\bar{H}^i$ are, condition (3.1.0) will hold. In other words the distribution of the price- cross effect matrices H^i would have to be *exceptionally highly skewed* for a per capita utility function to fail to exist.

Theorem 3.1. *Let $\bar{H}$ be any positive definite matrix and let $\bar{H}^i = \bar{H}^{-1/2}H^i\bar{H}^{-1/2}$, then the necessary and sufficient condition that a utility function exists is equivalent to finding conditions on $\bar{H}^i$ so that for all $p \neq 0$ and $q \neq \alpha p$:*

$$\mathcal{E}_i\left[\frac{q'\bar{H}^iq}{p'\bar{H}^ip} - 2\frac{(q'\bar{H}^ip)^2}{(p'\bar{H}^ip)^2}\right] + \left[\mathcal{E}_i\frac{q'\bar{H}^ip}{p'\bar{H}^ip}\right]^2 > 0\,. \tag{3.1.1}$$

Proof. The matrix $\bar{H}^{1/2}$ is not unique. There is a way to choose it so that $\bar{H}^{1/2}$ is symmetric, namely $\bar{H}^{1/2} = ED^{1/2}E'$ where E is the matrix of eigenvectors of $\bar{H}$ and D is the diagonal matrix whose diagonal is the eigenvalues of $\bar{H}$. For properties of E see proof of Theorem 3.3. Substituting $p = \bar{H}^{-1/2}\hat{p}$, $q = \bar{H}^{-1/2}\hat{q}$ into (3.1.0) and then relabeling $(\hat{p},\hat{q})$ as (p,q) in order not to have proliferation of symbols we obtain (3.1.1). ∎

Comment. If H^i are all close to $\bar{H} = \mathcal{E}_iH^i$, then $\bar{H}^i$ will be close to $\mathcal{E}_i\bar{H}^i = I$, the identity. If $\bar{H}^i = I$ for all i, condition (3.1.1) reduces to showing $p^2q^2 - (p'q)^2 > 0$ for all $p \neq 0$, $q \neq 0, p \neq \alpha q$ where p^2, q^2 denotes $p'p$, $q'p$ respectively. But the latter is always true because it is the same as $p^2q^2\sin^2\theta$ where θ is the angle between vectors p and q.

Theorem 3.2. *The necessary and sufficient condition (3.1.1) that a utility function exists is equivalent to finding conditions on $\bar{H}^i$ so that for all $\bar{p} \neq 0$, $\bar{q} \neq 0$, $\bar{p}'\bar{q} = 0$:*

$$\mathcal{E}_i\left[\frac{\bar{q}'\bar{H}^i\bar{q}}{\bar{p}'\bar{H}^i\bar{p}} - 2\frac{(\bar{q}'\bar{H}^i\bar{p})^2}{(\bar{p}'\bar{H}^i\bar{p})^2}\right] + \left[\mathcal{E}_i\frac{\bar{q}'\bar{H}^i\bar{p}}{\bar{p}'\bar{H}^i\bar{p}}\right]^2 > 0\,, \qquad \bar{p}'\bar{q} = 0\,. \tag{3.1.2}$$

Proof. The only difference between (3.1.1) and (3.1.2) is the requirement that $\bar{q}$ be orthogonal to $\bar{p}$. Because of homogeneity of (3.1.1) we can rescale p and q so that $p^2 = 1$ and $q^2 = 1$. Let θ be the angle between p and q so that $p'q = \cos\theta$. Note that the sign of the L.H.S. of (3.1.1) is the same if we replace q by $-q$ so that we need only consider $0 \leq \theta < \pi$. Therefore, the condition $q \neq \alpha p$ translates into $\sin\theta \neq 0$. We replace variables (p, q) in (3.1.1) by $\bar{p}, \bar{q}$, and θ where

$$p = \bar{p}, \;\; q = \bar{q}\sin\theta + \bar{p}\cos\theta, \;\; \cos\theta = p'q \quad \text{where} \;\; \sin\theta \neq 0\,. \tag{3.1.3}$$

It is easy to prove when $p^2 = 1, q^2 = 1$ that

$$\bar{p}'\bar{q} = 0, \quad \bar{q}^2 = 1, \quad \bar{p}^2 = 1\,. \tag{3.1.4}$$

Substitute (3.1.3) into (3.1.1). After much cancellation of terms and factoring out of the common factor $\sin^2\theta > 0$, we obtain (3.1.2). Because (3.1.2) is homogeneous in $\bar{p}$ and $\bar{q}$, we no longer require $\bar{p}^2 = 1$, $\bar{q}^2 = 1$. ∎

Comment. The last bracket expression of (3.1.2) is not likely to contribute much to the positivity of the L.H.S. For example, if $\bar{H} = \mathcal{E}_i H^i$ and all $\bar{H}^i = \mathcal{E}_i\bar{H}^i = I$, the identity, the last term would vanish because $\bar{p}'\bar{q} = 0$. Therefore, if we drop the second bracket, *it is sufficient* to only consider conditions on $\bar{H}^i$ that guarantee that the first bracket expression is positive. As an extremum or worst case scenario, we will look for conditions on $\bar{H}^i$ that will guarantee *for every i* that t_i, the corresponding pair of ith terms in the first bracket, is positive:

$$t_i = \frac{(\bar{q}'\bar{H}^i\bar{q})(\bar{p}'\bar{H}^i\bar{p}) - 2(\bar{q}'\bar{H}^i\bar{p})^2}{(\bar{p}'\bar{H}^i\bar{p})^2} > 0 \quad \text{for all } \bar{p}^2 \neq 0,\ \bar{q}^2 \neq 0 \quad \text{such that } \bar{p}'\bar{q} = 0\,. \tag{3.1.5}$$

To simplify the discussion, we assume all the eigenvalues of $\bar{H}^i$ are distinct. If not they could be made so by a slight perturbation of H^i.

Theorem 3.3, a sufficient condition. *A utility function exists if for each i:*

$$t_i = \frac{q'Dq}{p'Dp} - 2\frac{(q'Dp)^2}{(p'Dp)^2} > 0 \quad \textit{for all } p'q = 0,\ p^2 = 1,\ q^2 = 1 \tag{3.2.0}$$

where $D = D^i$ is the diagonal matrix whose diagonal elements are $d_1^i < \cdots < d_m^i$, the m distinct eigenvalues of $\bar{H}^i$.

Proof. All the eigenvalues of the positive-definite matrix $\bar{H}^i$ are positive. Let these be $d_k^i > 0$. To simplify the notation we will write $d_k = d_k^i$ and denote their corresponding eigenvectors by $E_k = E_k^i$ rescaled so that $(E_k)^2 = 1$. Then by definition

$$\bar{H}^i E_k = d_k \cdot I \cdot E_k \ ; \ \ E'_\ell \bar{H}^i = d_\ell \cdot E'_\ell \cdot I \ , \quad (k,\ell) = 1,\ldots,m \tag{3.2.1}$$

where I denotes the identity matrix. Therefore $E'_\ell \bar{H}^i E_k = d_k \cdot E'_\ell \cdot E_k = d_\ell \cdot E'_\ell \cdot E_k$. Since $d_k \neq d_\ell$ for all $\ell \neq k$, it follows that $E'_\ell E_k = 0$. Therefore the matrix of eigenvectors $E = (E_1, E_2, \ldots, E_m)$ is an *orthonormal* matrix, i.e., $E'E = I$, and

$$E' \bar{H}^i E = D \ . \tag{3.2.2}$$

Substituting $\bar{q} = Eq$, $\bar{p} = Ep$ into (3.1.5) and rescaling so that $p^2 = 1$, $q^2 = 1$, $p'q = 0$ we obtain (3.2.0). ∎

Theorem 3.3. *A utility function exists if for every i the diagonal matrix $D = D^i$ of eigenvalues of $\bar{H}^i$ has the property* $\min t_i > 0$ *where*

$$t_i = 1 - 2\frac{(\rho_i - 1)^2}{(\rho_i + 1)^2} > 0 \ , \tag{3.3.0}$$

where ρ_i is the ratio of the highest to lowest eigenvalues of $\bar{H}^i$, or equivalently when

$$\rho_i < 3 + 2\sqrt{2} \doteq 5.83 \tag{3.3.1}$$

Proof. Condition (3.3.1) follows by rewriting (3.3.0) as $(-\rho^2 + 6\rho - 1)/(\rho + 1)^2$ and determining the range of ρ where $y = -\rho^2 + 6\rho - 1 > 0$. To prove (3.3.0), we minimize t_i given by (3.2.0) subject to $p^2 = 1$, $q^2 = 1$, $p'q = 0$. For the p, q that yields this minimum, let

$$p'Dp = \alpha > 0 \quad \text{and} \quad q'Dq = \beta > 0 \ , \quad p'q = 0 \ . \tag{3.3.2}$$

Under the conditions $p'Dp = \alpha$ and $q'Dq = \beta$, minimizing t_i reduces to

$$\max(q'Dp)^2 \quad \text{subject to (3.3.2)} \ . \tag{3.3.3}$$

From the properties at the maximum, we will derive a relationship between $p'Dp$ and p^2, and $q'Dq$ and q^2 that will allow us to minimize t_i subject to $p'q = 0$, $p^2 = 1$, $q^2 = 1$, see (3.2.0).

Fact. $q'Dp \neq 0$ at the maximum. **Proof**: We assumed diagonal D satisfies $0 < d_2 < \cdots < d_m$. Given $\alpha > 0$ and $\beta > 0$, it is easy to find p and q satisfying (3.3.2) and $q'Dp \neq 0$. Therefore, because everything is bounded and continuous, a maximum exists with $(q'Dp)^2 > 0$. We can now form the Lagrangian

$$L = (q'Dp)^2 - \lambda(p'Dp) - \mu(q'Dq) - 2\nu(p'q) \tag{3.3.4}$$

and set $\partial L/\partial p = 0$ and $\partial L/\partial q = 0$, obtaining

$$(q'Dp) \cdot Dq = \lambda \cdot Dp + \nu q \,, \tag{3.3.5}$$

$$(q'Dp) \cdot Dp = \mu \cdot Dp + \nu p \,, \tag{3.3.6}$$

Fact: $\nu \neq 0$. **Proof:** Since $q'Dp \neq 0$, it follows $\nu \neq 0$ because $\nu = 0$ would imply by (3.3.5) that p and q are proportional and $p'q \neq 0$, contrary to hypothesis.

Fact: $\lambda > 0$, $\mu > 0$. This can be seen by multiplying (3.3.5) and (3.3.6) by p and q respectively:

$$(q'Dp)^2 = \lambda\alpha = \mu\beta \,. \tag{3.3.7}$$

Multiplying (3.3.5) and (3.3.6) by q and p respectively:

$$(q'Dp)\beta = \lambda(q'Dp) + \nu q^2 = (q'Dp)^3/\alpha + \nu q^2 \tag{3.3.8}$$

$$(q'Dp)\alpha = \mu(q'Dp) + \nu p^2 = (q'Dp)^3/\beta + \nu p^2 \,. \tag{3.3.9}$$

Hence

$$(q'Dp)\alpha\beta = (q'Dp)^3 + \alpha\nu q^2 \tag{3.4.0}$$

$$(q'Dp)\alpha\beta = (q'Dp)^3 + \beta\nu p^2 \tag{3.4.1}$$

implying

$$(p^2/\alpha) = (q^2/\beta) \quad \text{and} \quad \alpha = \beta \quad \text{when} \quad p^2 = q^2 = 1 \,. \tag{3.4.2}$$

We now rewrite (3.3.5) and (3.3.6)

$$[(q'Dp) \cdot D - \nu I]q = \lambda Dp \tag{3.4.3}$$

$$[(q'Dp) \cdot D - \nu I]p = \mu Dq \tag{3.4.4}$$

where I is the identity matrix. Solving (3.3.7) for λ and (3.3.8) for ν and substituting into (3.4.3):

$$\left\{(q'Dp) \cdot D - \frac{1}{\alpha q^2}\left[\alpha\beta(q'Dp) - (q'Dp)^3\right] \cdot I\right\}q = \frac{(q'Dp)^2}{\alpha} \cdot Dp \,, \tag{3.4.5}$$

and an analogous expression if we solve (3.3.7) for μ and (3.3.9) for ν and substitute into (3.4.4). We can now factor out $(q'Dp) \neq 0$, obtaining (3.4.6), and by analogy (3.4.7):

$$\left\{\alpha q^2 D - \left[\alpha\beta - (q'Dp)^2\right] I\right\}q = q^2(q'Dp) \cdot Dp \tag{3.4.6}$$

$$\left\{\beta p^2 D - \left[\alpha\beta - (q'Dp)^2\right] I\right\}p = p^2(q'Dp) \cdot Dq \tag{3.4.7}$$

where the expression in brackets, are the same in (3.4.6) and (3.4.7) since $\alpha q^2 = \beta p^2$. Noting that the product of diagonal matrices is commutative, multiply (3.4.7) by $q^2(q'Dp)D$ and interchange

on the L.H.S. the order of $q^2(q'Dp)D$ and the bracket term; finally substituting L.H.S. of (3.4.6) for $q^2(q'Dp) \cdot Dp$, we obtain

$$\{\alpha q^2 D - [\alpha\beta - (q'Dp)^2]I\}^2 q = p^2 q^2 (q'Dp)^2 \; D^2 q \, . \tag{3.5.0}$$

Expanding and rearranging terms and noting $(\alpha q^2)^2 = p^2 q^2 \alpha\beta$:

$$[p^2q^2\{\alpha\beta - (q'Dp)^2\}D^2 - 2(\alpha q^2)\{\alpha\beta - (q'Dp)^2\}D - \{\alpha\beta - (q'Dp)^2\}^2 I]q = 0 \, . \tag{3.5.1}$$

We observe that the common factor $\{\alpha\beta - (q'Dp)^2\} \neq 0$ by (3.4.0) or (3.4.1) and the fact that $\nu \neq 0$. Therefore factoring out $\{\alpha\beta - (q'Dp)^2\} \neq 0$ from (3.5.1), we obtain (3.5.2) and an analogous expression, (3.5.3) after noting $\alpha q^2 = \beta p^2$:

$$[p^2q^2D^2 - 2(\alpha q^2)D - (\alpha\beta - (qDp)^2)I]q = 0 \, , \tag{3.5.2}$$

$$[p^2q^2D^2 - 2(\alpha q^2)D - (\alpha\beta - (qDp)^2)I]p = 0 \, . \tag{3.5.3}$$

Vector relation (3.5.2) holds for each component q_k of q. Since $q^2 \neq 0$, one or more components $q_k \neq 0$. Let $q_k \neq 0$ for some k, then for this k:

$$(p^2q^2)d_k^2 - 2(\alpha q^2)d_k - [\alpha\beta - (q'Dp)^2] = 0 \, . \tag{3.5.4}$$

Let $y = (p^2q^2)x^2 - 2(\alpha q^2)x - [\alpha\beta - (q'Dp)^2]$ be a parabola expressing y as a function of x. Now $y = 0$ can only hold for at most two values of $x = d_k$, i.e., for say $k = 1$ and $k = m$. And from (3 [illegible] only *the same* $k = 1$ and $k = m$ can possibly have $p_k \neq 0$, the case of only one k being rul[illegible] out because say $k = 1$ only, then both $p_1 \neq 0, q_1 \neq 0$; and all other $p_i = 0, q_i = 0$. But this contradicts $p'q = 0$. Representing these two components of p and q in polar coordinates and noting $p'q = 0$:

$$p_1 = +\|p\|\cos\theta, \quad p_m = +\|p\|\sin\theta \, , \qquad \text{and } \; p_i = 0 \quad \text{for } \; 1 < i < m \, , \tag{3.6.0}$$

$$q_1 = -\|q\|\sin\theta, \quad q_m = +\|q\|\cos\theta \, , \qquad \text{and } \; q_i = 0 \quad \text{for } \; 1 < i < m \, , \tag{3.6.1}$$

subject to $p'Dp = \alpha$, $q'Dq = \beta$, $p'q = 0$. Therefore

$$\alpha = p'Dp = p^2[d_1\cos^2\theta + d_m\sin^2\theta] \tag{3.6.2}$$

$$\beta = q'Dq = q^2[d_1\sin^2\theta + d_m\cos^2\theta] \, . \tag{3.6.3}$$

Whence, noting $p^2/\alpha = q^2/\beta$, by (3.4.2),

$$d_1\cos^2\theta + d_m\sin^2\theta = d_1\sin^2\theta + d_m\cos^2\theta \, , \tag{3.6.4}$$

$$(d_m - d_1)[\cos^2\theta - \sin^2\theta] = 0 \, , \tag{3.6.5}$$

and therefore since $d_m \neq d_1$,

$$\cos^2\theta = \tfrac{1}{2}\,, \qquad\qquad \sin^2\theta = \tfrac{1}{2}\,, \tag{3.6.6}$$

$$\alpha = p^2[d_1 + d_m]/2, \qquad \beta = q^2[d_1 + d_m]/2\,, \tag{3.6.7}$$

$$(qDp)^2 = p^2q^2\sin^2\theta\cos^2\theta(d_m - d_1)^2 = \frac{\alpha\beta(d_m - d_1)^2}{(d_1 + d_m)^2}\,. \tag{3.6.8}$$

Recalling $\alpha = \beta$ from (3.4.2),

$$t_i = \frac{\beta}{\alpha} - 2\frac{\alpha\beta(d_m - d_1)^2}{\alpha^2(d_m + d_1)^2} = \left[1 - 2\frac{(d_m - d_1)^2}{(d_m + d_1)^2}\right]. \tag{3.6.9}$$

We left open the question, which pair of the distinct eigenvalues $d_1 < d_2 < \cdots < d_m$ to select; we show now that the best choice for $\min t_i$ is in fact the ones we chose, namely d_1 and d_m. Let $j > i$ and $y = (d_j - d_i)/(d_j + d_i)$, then $\partial y/\partial d_j = 2d_i/(d_j + d_i)^2 > 0$ and $\partial y/\partial d_i = -2d_j/(d_j + d_i)^2 < 0$. Therefore the ratio increases as $d_j \to d_m$ and also as $d_i \to d_1$. Hence the lowest t_i is obtained for $i = 1$, and $j = m$.

From (3.6.0) and (3.6.1), optimal $p_1 = \cos(\pi/4)$, $p_m = \sin(\pi/4)$, and $p_i = 0$ for $1 < i < m$; and also $q_1 = -\sin(\pi/4)$, $q_m = cos(\pi/4)$, and $q_i = 0$ for $1 < i < m$. Therefore the minimal solution to (3.2.0) is

$$\min\ t_i = \left[1 - 2\frac{(d_m - d_1)^2}{(d_m + d_1)^2}\right] \tag{3.7.0}$$

Let $\rho = d_m/d_1$ be the ratio of the highest to lowest eigenvalue, then

$$\min\ t_i = 1 - 2\frac{(\rho - 1)^2}{(\rho + 1)^2} = (-\rho^2 + 6\rho - 1)/(\rho + 1)^2 \tag{3.7.1}$$

which is positive for all ρ in the range

$$(3 + 2\sqrt{2})^{-1} < \rho < 3 + 2\sqrt{2} \doteq 5.83\,. \quad \blacksquare \tag{3.7.2}$$

Comment. In order for a utility function to exist it is sufficient that $\mathcal{E}_i t_i > 0$. Therefore the condition that $\min t_i > 0$ for each i is a worse case scenario that is far too stringent. We know for case of a population of two people that it is possible for $\mathcal{E}_i t_i < 0$, and this is true even if the term $[\mathcal{E}_i(q'\bar{H}^i p)/(p'H^i p)]^2$ is added to t_i, see counter example of Theorem 2.5. In a large population such as U.S. we would expect the set of representative ellipsoids $p'\bar{H}^i p = constant$ in the urn which can be rotated into one another to have to some extent randomly distributed orientations of their axes. If we rotate all such ellipsoids to H^i for the lowest i, say H, the effect is to rotate the p and q to a random position. Therefore, we need to consider for each such H, the average or expected value of

$$f(p,q) = \mathcal{E}_{p,q}\left[\frac{q'Hq}{p'Hp} - 2\left(\frac{q'Hp}{p'Hp}\right)^2\right], \qquad p^2 = 1, \quad q^2 = 1, \quad p'q = 0\,, \tag{3.7.3}$$

where p and q are "randomly" distributed in some way. We, of course, do not know what is the true distribution of orientations of the axes of the ellipsoids $p'H^ip = constant$ that can be rotated into one another. We will prove a theorem when these orientations are uniformly distributed about the origin in R^m. While this assumption of "uniformly distributed" is not realistic, the purpose of Theorem 3.5 is to illustrate that the more the orientations vary relative to one another, the higher the bound $\rho < 3 + 2\sqrt{2} + \Delta$ can be where $\Delta \to +\infty$.

Theorem 3.5. *If the orientations of the axes of the ellipsoids $p'H^ip = constant$ that can be rotated into any particular $H^i = H$ are uniformly distributed (or nearly so), then $\mathcal{E}_{p,q}f(p,q) > 0$ where $f(p,q)$ is defined by (3.7.3); moreover $\mathcal{E}_it_i > 0$, implying that the expected per capita utility function exists.*

Proof. The proof consists in partitioning the set

$$S(p,q) = \{p,q|p^2 = 1,\ q^2 = 1,\ p'q = 0\} \tag{3.8.0}$$

of all admissible (p,q) into equally probable subsets $T(R)$, each subset obtained from $T(I)$ by a general rotation R in R^m. We will show that each $T(R)$ has the property, that the average value of $f(p,q)$ found by integration over $T(R)$ is unity. It then follows that the integration of $f(p,q)$ over $S(p,q)$ will also be unity. As noted one of the subsets $T(R)$ is the set $T(I)$ which is defined to be the set of all $\bar{p} = (\cos\phi, \sin\phi, 0, \ldots, 0)'$ and $\bar{q} = (-\sin\phi, \cos\phi, 0, \ldots, 0)'$ for $0 \le \phi \le 2\pi$.

This "reduces" the subproblem to considering, in place of $H, \bar{p}, \bar{q}$, the truncated matrix and vectors:

$$\tilde{H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix}, \quad \tilde{p} = (\cos\phi, \sin\phi),\ \ \tilde{q} = (-\sin\phi, \cos\phi)\ . \tag{3.8.1}$$

Let the matrix of eigenvectors of $\tilde{H}$ be $\tilde{E}$ so that $\tilde{E}'\tilde{H}\tilde{E} = \hat{D}$. The diagonal matrix whose elements are the eigenvalues of $\tilde{H}$, namely $\hat{D}(1,1) = d_1 > 0$ and $\hat{D}(2,2) = d_2 > 0$. The matrix $\tilde{E}$ is orthonormal, hence we can rotate $\tilde{p}$ and $\tilde{q}$ in R^2 space by $p = \tilde{E}\hat{p}$ and $\hat{q} = \tilde{E}\hat{q}$ and $\phi = \theta + \alpha$ so that the axes of the ellipsoid $\tilde{p}'\tilde{H}\tilde{p} = constant$ are parallel to the coordinate axes. After this rotation, our problem reduces to showing for $T(I)$:

$$\frac{2}{\pi}\int_0^{\pi/2}\left[\frac{d_1\sin^2\theta + d_2\cos^2\theta}{d_1\cos^2\theta + d_2\sin^2\theta} - 2\frac{(d_2 - d_1)^2\sin^2\theta\cos^2\theta}{[d_1\cos^2\theta + d_2\sin^2\theta]^2}\right]d\theta = 1\ . \tag{3.8.2}$$

where $\hat{p} = (\cos\theta, \sin\theta), \hat{q} = (-\sin\theta, \cos\theta)$. Note we claim that relation (3.8.2) holds *independent* of what the diagonal elements of $\hat{D}$ happen to be.

We will prove (3.8.2) in a moment. We can obtain the subset $T(R)$, by rotating the subset $T(I)$ by a general rotation R^{-1} about the origin in R^m space. Conversely we can rotate $T(k)$ into $T(I)$ by $p = R\bar{p}$, $q = R\bar{q}$, where R is an orthonormal matrix. The effect is to replace H by $R'HR$ and carry out the integration over $T(I)$ with respect to $R'HR$, instead of H, again obtaining unity. Since each rotation R is equally probable, this means $\mathcal{E}_{p,q}f(p,q) = 1$ for $(p,q) \in S(p,q)$.

We now show (3.8.2) is true. Substituting $x = \cos^2\theta + \rho\sin^2\theta$, and letting $d_2 = \rho d_1$, and $Q = -\rho + (\rho+1)x - x^2$, (3.8.2) reduces to proving

$$\begin{aligned}
&\frac{2}{\pi}\int_1^{\rho}\Big[\frac{\rho+1-x}{2xQ^{1/2}} - \frac{Q^{1/2}}{x^2}\Big]dx \\
&\quad = \frac{2}{\pi}\Big[\frac{\rho+1}{2}\int_1^{\rho}\frac{dx}{xQ^{1/2}} - \frac{1}{2}\int_1^{\rho}\frac{dx}{Q^{1/2}} - \int_1^{\rho}\frac{Q^{1/2}dx}{x^2}\Big] = 1
\end{aligned} \tag{3.8.3}$$

Denoting $a + bx + cx^2 = Q = -\rho + (\rho+1)x - x^2$ and noting $(b^2-4ac)^{1/2} = \rho - 1$, we obtain from a table of standard integrals:

$$\begin{aligned}
\int_1^{\rho}\frac{dx}{x\mathrm{Q}^{1/2}} &= \frac{1}{(-a)^{1/2}}\sin^{-1}\left(\frac{bx+2a}{x(b^2-4ac)^{1/2}}\right) \\
&= \frac{1}{\rho^{1/2}}\sin^{-1}\left(\frac{(\rho+1)x-2\rho}{x(\rho-1)}\right)\Bigg|_{x=1}^{\rho} = \frac{\pi}{\rho^{1/2}}
\end{aligned} \tag{3.8.4}$$

$$\begin{aligned}
\int_1^{\rho}\frac{dx}{\mathrm{Q}^{1/2}} &= \frac{1}{(-c)^{1/2}}\sin^{-1}\left(\frac{-2cx-b}{(b^2-4ac)^{1/2}}\right) \\
&= \sin^{-1}\left(\frac{2x-(\rho+1)}{\rho-1}\right)\Bigg|_{x=1}^{\rho} = \pi
\end{aligned} \tag{3.8.5}$$

$$\begin{aligned}
\int_1^{\rho}\frac{\mathrm{Q}^{1/2}}{x^2}dx &= -\frac{\mathrm{Q}^{1/2}}{x}\Bigg|_{x=1}^{\rho} + \frac{b}{2}\int_1^{\rho}\frac{dx}{x\mathrm{Q}^{1/2}} + c\int_1^{\rho}\frac{dx}{\mathrm{Q}^{1/2}} \\
&= 0 + \frac{\rho+1}{2}\Big(\frac{\pi}{\rho^{1/2}}\Big) - \pi
\end{aligned} \tag{3.8.6}$$

Substituting these evaluations into (3.8.3), yields $\mathcal{E}_{p,q}f(p,q) = 1$, implying $\mathcal{E}_i t_i = 1 > 0$, which is sufficient for the expected per capita utility function to exist. ∎

In this part we explored how close the price cross-effect matrices H^i must be to a typical $\bar{H}$ to guarantee integrability. One measure of closeness is the ratio ρ_i of the highest to lowest eigenvalues of $\bar{H}^i = \bar{H}^{-1/2}H^i\bar{H}^{-1/2}$. It is sufficient if $\rho_i < 3 + 2\sqrt{2}$ for all i. Another measure is how spread out is the distribution of orientations of the axes of the ellipsoids $p'H^ip = constant_i$, the more evenly spread the better. We found that the bound on ρ_i is higher the more the tilts of the axes of the ellipsoids $p'\bar{H}^ip = constant_i$ are uniformly distributed relative to one another. On the other hand, if individuals i in the population tend to be highly polarized as to their consumption tastes, as in the example of Theorem 2.5 with $\rho_1 = \rho_2 = 91$ a per capita utility function may not exist.

PART IV:
INTEGRALITY OF THE MULTIPLE-PERIOD EQUILIBRIUM MODEL

In Part III we established that the expected per capita for period t *over the income range of interest* is integrable when the ratio ρ_i of highest to lowest eigenvalue of $\bar{H}^i$ is less than $3 + \sqrt{8}$ for all i, or less than a higher bound for ρ_i if the axes of ellipsoids $p'H^ip = const.$ are randomly disposed to one another. From now on we assume that a utility function for each period exists, and consequently an inverse demand function exists except for a scale factor to be determined. This does not mean, however, in the context of a time-staged model for the whole economy, that there exists an objective function which, when maximized subject to physical flow constraints, implies the additional equilibrium conditions involving prices that must hold between the producer/investor and final consumer. It depends on how the numeraire for prices in the rate-of-return formulas are defined. It turns out that when the numeraire for normalizing prices is suitably defined, a concave objective function for the economy exists.

Aggregate quantities corresponding to per capita quantities are denoted by bold face letters $\mathbf{X}_t, \mathbf{S}_t, \mathbf{I}_t, \mathbf{U}_t, \mathbf{Z}_t$. They are obtained from $\bar{X}_t, \bar{S}_t, \bar{I}_t, \bar{U}_t, Z_t$ by multiplying by P_t, the size of population in period t.

The mathematical formulation of the multi-period model is along the classical lines of an Arrow-Debreu [1,5] or Scarf [17] equilibrium model, with no surprises except perhaps for the interpretation of the profitability constraints of investors as rate-of-return formulae for selecting among different investment possibilities. A typical production/investment activity j in period t has a column consisting of three sets of fixed coefficients $[B_t(j),\ -A_t(j), -D_t(j)]'$ per unit level of activity where $B_t(j)$ is the input/output vector of capacities, resources, and flows of all items (goods) needed for production and capacity formation in period t *except final consumer items*; $A_t(j)$ is the output/input vector of final consumer items in period t; and $D_t(j)$ is the output vector of capacities, resources, and intermediate goods left over or produced in period t for period $t+1$. The model is then defined by five sets of relations numbered (4.1.0) through (4.5.0). The first two we refer to as the primal or physical flow constraints

In (4.1.0) below, $\mathbf{Y}_t \geq 0$ is the vector of aggregate production and investment levels to be determined in period t. In words, (4.1.0) states that the capacity, resources, intermediate items $B_t\mathbf{Y}_t$, required for production and capacity formation at levels $\mathbf{Y}_t$ cannot exceed the amount of these items $D_{t-1}\mathbf{Y}_{t-1}$ left over or produced by period $t-1$ activities for period t plus the vector k_t of these items exogenously supplied:

$$B_t\mathbf{Y}_t \leq D_{t-1}\mathbf{Y}_{t-1} + k_t\ ; \qquad\qquad \text{corresp. dual} \quad \sigma_t \geq 0\ . \tag{4.1.0}$$

The vector of dual prices corresponding to (4.1.0) is denoted by $\sigma_t \geq 0$, and the slack vector which turns (4.1.0) into an equation is denoted by $\hat{\sigma}_t \geq 0$. According to the theory of Arrow-Debreu,

production levels $\mathbf{Y}_t$ and prices σ_t will adjust until at equilibrium the complementary slackness conditions $\hat{\sigma}_t'\sigma_t = 0$ hold.

In (4.2.0) below, $\mathbf{X}_t \geq 0$ is the vector of aggregate final consumption in period t measured in physical units. In words, (4.2.0) states that consumption cannot exceed $A_t\mathbf{Y}_t$, the net output from production after investment less, f_t, any fixed demand (like government plus any minimum floor provided to the final consumer and not paid for by income for consumption):

$$-A_t\mathbf{Y}_t + \mathbf{X}_t \leq -f_t \; ; \qquad \text{corresp. dual} \quad \pi_t \geq 0 \, . \tag{4.2.0}$$

The corresponding vector of normalized discounted dual prices is denoted by $\pi_t \geq 0$, and the slack vector which turns (4.2.0) into an equation is denoted by $\hat{\pi}_t \geq 0$. At equilibrium $\hat{\pi}_t'\pi_t = 0$.

Dual Constraints: We assume that a numeraire has been selected in each period relative to which prices for various goods are measured. If so we say the prices $\bar{\pi}$ are "normalized". Prices $\sigma_t \geq 0$ and $\pi_t \geq 0$ for the dual constraints are defined to be *discounted period t prices*; moreover π_t is defined to be discounted *normalized* period t prices. Thus $\pi_t = \delta^{t-1}\,\bar{\pi}_t$ where the discount factor is δ^{t-1} and $\bar{\pi}_t$ is the vector of normalized prices. The vector of unnormalized period t prices of final consumer items is denoted by p_t. We have no need for a symbol for prices on capacity, reserves, and intermediate goods relative to unnormalized prices p_t but prices on these items relative to normalized prices $\bar{\pi}_t$ are denoted by $\bar{\sigma}_t$.

Relation (4.3.0) below, which we will derive in the comment, states that investors must receive at least their minimum rate of return $r = \delta^{-1} - 1$ or they won't invest:

$$-B_t'\sigma_t + A_t'\pi_t \leq -D_t'\sigma_{t+1} \; ; \qquad \text{corresp. primal} \quad \mathbf{Y}_t \geq 0 \, . \tag{4.3.0}$$

The slack vector which turns (4.3.0) into an equation is denoted by $\hat{\mathbf{Y}}_t \geq 0$. It is in 1 to 1 correspondence with primal variables $\mathbf{Y}_t \geq 0$. At equilibrium $\hat{\mathbf{Y}}_t'\mathbf{Y}_t = 0$.

Comment: Typically, the *rate-of-return* formula for a producer/investor j in period t is formed by "pricing-out" the input-output vector of physical flows per unit level of investment activity j, namely $[B_t(j), -A_t(j), -D_t(j)]'$, and then looking for a discount factor $\bar{\delta}$ so that the discounted cash flows is zero:

$$-\bar{\delta}^{t-1}\bar{\sigma}_t'B_t(j) + \bar{\delta}^{t-1}\bar{\pi}_t'A(j) + \bar{\delta}^t\bar{\sigma}_{t+1}'D_t(j) = 0. \tag{4.3.1}$$

Note the use of *normalized prices* $\bar{\pi}_t$ (and $\bar{\sigma}_t$ relative to $\bar{\pi}_t$), and not unnormalized prices p_t. This makes the rate-of-return formula *inflation free* meaning that the investor decides what his minimum rate of return r should be without multiplying it by some factor for future inflation. It is assumed that an investment must have a rate of return not less than a minimum rate r, or a d unt factor $\bar{\delta} \leq \delta = (1+r)^{-1}$. This implies that investors will not consider investment opportunity j if its $\bar{\delta} > \delta$. On the other hand if its $\bar{\delta} < \delta$, the investment is profitable and the level $\mathbf{Y}_t(j)$ will tend in the "real world" to increase indefinitely. This puts pressure on capacities and resources that are in

short supply, thus forcing prices and $\bar{\delta}$ up, until at "equilibrium" $\bar{\delta}^{-1} = 1 + r$. We express these conditions by defining $\hat{\mathbf{Y}}_t(t) \geq 0$ by

$$-\delta^{t-1}\bar{\sigma}_t' B_t(j) \; + \; \delta^{t-1}\bar{\pi}_t' A(j) \; + \; \delta^t \bar{\sigma}_{t+1}' D_t(j) \; + \; \hat{\mathbf{Y}}_t(j) = 0 \tag{4.3.2}$$

and requiring $\hat{\mathbf{Y}}_t(j) \cdot \mathbf{Y}_t(j) = 0$. Substituting $\delta^{t-1}\bar{\sigma}_t = \sigma_t$, $\delta^{t-1}\bar{\pi}_t = \pi_t$, $\delta^t\bar{\sigma}_{t+1} = \sigma_{t+1}$ and dropping the slack vector $\hat{\mathbf{Y}}(j) \geq 0$, this relation is the same as (4.3.0). End of comment.

In (4.4.0) below, $F_t(\mathbf{X}_t)$ is the *inverse of the expected aggregate demand function* which we assume now exists except for a scale factor to be determined that generates normalized period t prices $\bar{\pi}_t$ required by the rate-of- return formula (4.3.0):

$$-\pi_t + \delta^{t-1} F_t(\mathbf{X}_t) \leq 0\,; \qquad\qquad \text{corresp. primal} \quad \mathbf{X}_t \geq 0\,. \tag{4.4.0}$$

The corresponding set of primal variables is $\mathbf{X}_t \geq 0$. When $\mathbf{X}_t > 0$, which is usually the case, this relation becomes an equation. The slack vector that turns (4.4.0) into an equation is denoted by $\hat{\mathbf{X}}_t \geq 0$. At equilibrium $\hat{\mathbf{X}}_t'\mathbf{X}_t = 0$. The case when the inverse demand function does not exist will be commented on in a moment.

The Complementary Slackness Conditions are that all variables be nonnegative and

$$\hat{\sigma}_t'\sigma_t = 0, \quad \hat{\pi}_t'\pi_t = 0, \quad \hat{\mathbf{Y}}_t'\mathbf{Y}_t = 0, \quad \hat{\mathbf{X}}_t'\mathbf{X}_t = 0. \tag{4.5.0}$$

The model has $t = 1, 2, \ldots, T$ periods. For period $t = 1$, the term $D_0 Y_0$ is omitted in relation (4.1.0). For period $t = T$, the term $-\sigma_{T+1}' D_T'$ is omitted in relation (4.3.0).

This completes the mathematical statement of the time-staged model. The remainder of Part IV is concerned with deriving the functional form of the *inverse* aggregate demand function $F_t(\mathbf{X}_t)$, and the utility function $\mathbf{U}(\mathbf{X}_1, \ldots, \mathbf{X}_t)$ for the full economy when it exists. Before doing so we note that if the economy were driven by a utility function of the form $\mathbf{U} = \Sigma\delta^{t-1}\mathbf{U}_t(\mathbf{X}_t)$, the Kuhn-Tucker conditions derived by maximizing $\mathbf{U}$ subject to primal physical-flow conditions (4.1.0) and (4.2.0) would give rise to conditions (4.3.0), (4.4.0) and (4.5.0) where $F_t(\mathbf{X}_t) = \partial\mathbf{U}_t/\partial\mathbf{X}_t$, see reference [12]. If the latter conditions hold for all t we say the vector functions $F_t(\mathbf{X}_t)$ in the context of the full model are *integrable*. If not, we say the model is non- integrable and no utility function for the economy exists.

The dynamic equilibrium problem is well defined even for the case that the inverse demand function for each period does not exist. In place of (4.4.0), we could state (a) the direct aggregate demand function (4.6.0) below replacing $\mathbf{I}_t$ by $\pi_t'\mathbf{X}_t$, and (b) the condition prices must satisfy when they are normalized and discounted. It is outside the scope of this paper to discuss whether or not an equilibrium solution might exist in this case.

Value of Endowments

Since the ***aggregate*** demand function expresses consumption $\mathbf{X}_t = P_t \cdot \bar{X}_t$ as a function of aggregate income $\mathbf{I}_t = P_t \cdot \bar{I}_t$ and prices, we reexpress Theorem 1.5:

$$\mathbf{S}_t - \mathbf{X}_t = (\pi_t' \mathbf{S}_t - \mathbf{I}_t) \xi_i (\pi_t' H^i \pi_t)^{-1} \cdot H^i \pi_t \,, \qquad P_t \cdot \bar{I}_t^* \leq \mathbf{I}_t \leq P_t \cdot \bar{I}_t^{**} \,, \tag{4.6.0}$$

where P_t is total size of the population in period t, $\mathbf{S}_t = P_t \cdot \bar{S}_t$, and $\pi_t = \delta^{t-1} \bar{\pi}_t$ are discounted normalized period t prices $\bar{\pi}_t$; aggregate income $\mathbf{I}_t$ is redefined to be aggregate income measured in terms of π_t instead of p_t. Before we place $F_t(\mathbf{X}_t)$, the inverse of this demand function, into the model, we must first relate its $\mathbf{I}_t$ to the value of endowments. This is important because the model does not provide any detail about who owns the endowments and therefore no detail about how rents, wages, dividends, royalties, interest on loans, taxes, government doles, etc., get transferred to the final consumers for consumption. We therefore need to be assured that the model nevertheless implicitly provides a mechanism whereby the *total value* of endowments used for consumption are in fact transferred.

Theorem 4.1. *The value of endowments $\mathbf{I}_t$ used to produce consumption $\mathbf{X}_t$ in period t is exactly equal to $\pi_t' \mathbf{X}_t$, the attained level of aggregate income used for consumption in period t.*

Proof. In terms of the prices of the model, the value of endowments available to period t is $\sigma_t'(D_{t-1} \mathbf{Y}_{t-1} + k_t)$. If we subtract off the value used for fixed consumption $\pi_t' f_t$, less $\sigma_{t+1}' D_t \mathbf{Y}_t$ the value passed down to period $t+1$, the net by definition is $\mathbf{I}_t$:

$$\mathbf{I}_t = \sigma_t'(D_{t-1} \mathbf{Y}_{t-1} + k_t) - \pi_t' f_t - \sigma_{t+1}' D_t \mathbf{Y}_t \tag{4.6.1}$$

$$= \sigma_t' B_t \mathbf{Y}_t - \sigma_{t+1}' D_t \mathbf{Y}_t - \pi_t' f_t \tag{4.6.2}$$

$$= \pi_t'(A_t \mathbf{Y}_t - f_t) \tag{4.6.3}$$

where (4.6.2) follows from (4.1.0) and (4.5.0), and (4.6.3) follows from (4.3.0) and (4.5.0). Therefore by (4.2.0) and (4.5.0):

$$\mathbf{I}_t = \pi_t' \mathbf{X}_t \,. \quad \blacksquare \tag{4.6.4}$$

Thus, independent of the choice of the demand function, the primal conditions (4.1.0), (4.2.0), the profitability conditions (4.3.0) and the complementarity conditions (4.5.0) imply that $\mathbf{I}_t = \pi_t' \mathbf{X}_t$ holds.

Equivalent Concave Program

Having given the conditions that an expected per capita utility function and inverse of the expected per capita demand function exist, *we now assume these exist* and set aggregate $\mathbf{X}_t = P_t \bar{X}_t$, $\mathbf{S}_t = P_t \bar{S}$, $\mathbf{U}_t = P_t \bar{U}_t$, $\mathbf{I}_t = P_t \cdot \bar{I}_t$ where P_t is the size of the population in period t. Let

$$p = G^{-1}(v) \quad \text{where } v = \bar{S}_t - \bar{X}_t \tag{4.7.0}$$

It follows from (2.15.3), after rescaling $-\bar{U}_t(\bar{X}_t) = Z$ for population size,

$$\mathbf{U}_t(\mathbf{X}_t) = -P_t \cdot \mathcal{G}_i(p'H^i p)^{-1/2} \tag{4.7.1}$$

where $\mathcal{G}_i$ denotes *geometric mean* of $(p'H^i p)^{-1/2}$ for H^i, $i = (1, \ldots, n)$, in the population urn.

Theorem 4.2. *The equilibrium problem is equivalent to solving the concave program*

$$\max \mathbf{U}(\mathbf{X}) = \sum_{t=1}^{T} \delta^{t-1} \mathbf{U}_t(\mathbf{X}_t) \tag{4.8.0}$$

subject to $(\mathbf{X}_t, \mathbf{Y}_t) \geq 0$ *and the primal flow conditions* $t = 1, \ldots, T$:

$$B_t \mathbf{Y}_t \leq D_{t-1}\mathbf{Y}_{t-1} + k_t \tag{4.8.1}$$

$$-A_t \mathbf{Y}_t + \mathbf{X}_t \leq f_t \tag{4.8.2}$$

providing (i) the primal problem is feasible, (ii) the geometric mean of $(p_t' H^i p_t)^{1/2}$ *for* $t = 1, 2, \ldots, n$ *is used as a numeraire for normalizing prices* p_t*, and (iii) the aggregate income* $\mathbf{I}_t = \pi_t' \mathbf{X}_t$ *associated with the optimal solution satisfies* $P_t \cdot \bar{I}^* \leq \mathbf{I}_t \leq P_t \cdot \bar{I}^{**}$*, and (iv)* $\mathbf{X}_t > 0$*. Under these conditions an equilibrium solution exists.*

Proof. The condition that $\mathbf{I}_t$ is bounded between certain lower and upper limits is our way of saying that the demand function of all individuals j is of the form $S^j - X^j = (\pi S^j - I)(\pi' H^j \pi)^{-1} \cdot H^j \pi$, which would not be the case if their budgets I were extremely low or extremely high. The equilibrium problem obviously has no solution if the primal problem is infeasible. Here in Part IV, we assume conditions of Theorem 3.3 hold so that a utility function for each period exists. By definition of a utility function, $\mathbf{U}_t(\mathbf{X}_t)$ are homogeneous functions of degree 1 in $v = \mathbf{S}_t - \mathbf{X}_t$ which are strictly concave in v except along rays with $v = 0$ as origin. The concave program under these conditions has a finite optimal solution. The Kuhn-Tucker conditions for optimality turn out to be the same as the dual constraints of the equilibrium problem, (4.3.0), (4.4.0), and (4.5.0).

If $\mathbf{X}_t > 0$, then in the equilibrium model, by (2.7.1) and (2.15.3):

$$\bar{\pi}_t = F_t(\mathbf{X}_t) = \partial \mathbf{U}_t / \partial \mathbf{X}_t \tag{4.9.0}$$

$$= \partial Z / \partial v = Z \cdot p \tag{4.9.1}$$

$$= p / \mathcal{G}_i (p' H^i p)^{1/2} \tag{4.9.2}$$

$$= p_t / \mathcal{G}_i (p_t' H^i p_t)^{1/2} \ . \tag{4.9.3}$$

The denominator $\mathcal{G}_i(p_t' H^i p_t)^{1/2}$ may be viewed as a *numeraire* for normalizing period t prices p_t.

Therefore if a feasible solution to the primal exists, an optimal feasible solution exists that satisfies the Kuhn-Tucker optimality conditions, which is the same as saying an equilibrium solution exists. The strict concavity implies that the values of $\mathbf{X}_t$ and π_t are unique; those of $\mathbf{Y}_t$ and σ_t need not be unique. ∎

Comment. The above way of normalizing p is different from the conventional one $p/\bar{e}'p$ where $\bar{e}' = (1/m \cdots 1/m)$ and m is the number of components in the vector p. Later on we will present evidence why this way of normalizing p_t is just as satisfactory if not superior.

Theorem 4.3. *The existence of a utility function* $\mathbf{U}(\mathbf{X})$ *depends on how the numeraire for period t prices used by the investor/producers in the rate-of-return formulas is defined.*

Proof. It is sufficient to demonstrate this when $H^i = \bar{H}$ for all i. In this case $G(\pi_t) = (\pi_t' \bar{H} \pi_t)^{-1} \cdot \bar{H} \pi_t$ and the expected aggregate demand function (4.6.0) simplifies to:

$$v = \bar{S}_t - \bar{X}_t = (\pi_t \bar{S}_t - I_t)(\pi_t' \bar{H} \pi_t)^{-1} \cdot \bar{H} \pi_t \, . \tag{4.10.0}$$

Dropping the income factor and rescaling π_t as p, then for this special case:

$$v = (p' \bar{H} p)^{-1} \cdot \bar{H} p \, , \tag{4.11.1}$$

$$p = (v' M v)^{-1} \cdot M v \, , \qquad \text{where } M = \bar{H}^{-1} \, . \tag{4.11.2}$$

By (4.7.1) and (4.9.2), $-\mathbf{U}_t$ simplifies in this case to

$$-\bar{U}_t = Z = (p' \bar{H} p)^{-1/2} = (v' M v)^{1/2} \tag{4.11.3}$$

$$= [(\bar{S}_t - \bar{X}_t)' M (\bar{S}_t - \bar{X}_t)]^{1/2} \tag{4.11.4}$$

By (4.9.0), (4.9.1), (4.11.3) and (4.11.2), π simplifies to

$$\pi = \delta^{t-1} \bar{\pi} = \delta^{t-1} \mathbf{Z} \, p = \delta^{t-1} M v / (v' M v)^{1/2} \tag{4.11.5}$$

It is easy to show

$$(\bar{\pi}' \bar{H} \bar{\pi})^{1/2} = 1 \, . \tag{4.11.6}$$

and therefore the implied numeraire for normalizing prices is $(p_t \bar{H} p_t)^{1/2}$.

Because $\bar{I}_t = \pi_t' \bar{X}_t$, the right-hand side of (4.10.0) is a homogeneous function in π of degree 0, implying that the *inverse demand function* $F_t(\mathbf{X}_t)$, that expresses π_t in terms of $\mathbf{X}_t = P_t \cdot \bar{X}_t$, while not unique, can be determined uniquely except for a scale factor. Note in Theorem 4.2 we choose one particular way, see (4.9.3). When $\mathbf{X}_t > 0$, relation (4.4.0) becomes an equation $\pi_t = \delta^{t-1} F_t(\mathbf{X}_t)$ where $F_t(\mathbf{X}_t) = \bar{\pi}_t$ is defined to be *normalized* period t prices. We therefore must choose the proportionality factor so that $F_t(\mathbf{X}_t)$ automatically generates normalized prices of period t before discounting.

The numeraire for normalizing prices, however, can be chosen in more than one way. If investors calculate their rate of return based on period t prices p_t normalized by $\bar{\pi}_t = p_t / (p_t' \bar{H} p_t)^{1/2}$, for the special case $H^i = \bar{H}$ for all i, then for this per capita demand function simplification (4.10.0), the inverse per capita demand function as we have shown can be stated explicitly and is proportional to $M(\bar{S}_t - \bar{X}_t)$ or $M(\mathbf{S}_t - \mathbf{X}_t)$, see (4.11.2). Under this definition of normalization, $F_t(\mathbf{X}_t)$ satisfies for $\mathbf{X}_t > 0$, $M = \bar{H}^{-1}$

$$F_t(\mathbf{X}_t) = \frac{1}{[(\mathbf{S}_t - \mathbf{X}_t)' M (\mathbf{S}_t - \mathbf{X}_t)]^{1/2}} \cdot M(\mathbf{S}_t - \mathbf{X}_t), \tag{4.12.0}$$

which is the same as (4.11.5) before discounting. Note in this case, $\bar{\pi}'\bar{H}\bar{\pi} \equiv 1$ for all $\bar{\pi} = F_t(\mathbf{X}_t)$ since $[F_t(\mathbf{X}_t)'\bar{H}F_t(\mathbf{X}_t]^{1/2} \equiv 1$ for all $\mathbf{X}_t$. This way of normalizing $F_t(\mathbf{X}_t)$ *does depend* on the scaling of M and we therefore would like the scaling of M to be such that base-year prices $p_0 = (1,\dots,1)'$ remain the same before and after normalization. This is why earlier, we required $\bar{H}$ to satisfy $1 = p_0'\bar{H}p_0 = \sum_k \sum_\ell \bar{H}_{k\ell}$, see (1.1.1).

On the other hand, if investors calculate their rate of return based on the vector of prices p_t normalized within each period so that their average price is unity, i.e. normalized by $\bar{\pi}_t = p_t/\bar{e}'p_t$ where $\bar{e}' = (1/m,\dots,1/m)$, then the scale factor must be chosen so that normalized prices satisfy $\bar{e}'\bar{\pi} = 1$. $F_t(\mathbf{X}_t)$ under this definition of normalization satisfies for the special case $H^i = \bar{H}$ for all i:

$$F_t(\mathbf{X}_t) = \frac{1}{\bar{e}'M(\mathbf{S}_t - \mathbf{X}_t)} \cdot M(\mathbf{S}_t - \mathbf{X}_t), \qquad \text{for } \mathbf{X}_t > 0\,,\ \ M = H^{-1}. \tag{4.13.0}$$

Note $\bar{e}'F_t(\mathbf{X}_t) \equiv 1$ for all $\mathbf{X}_t$. Moreover under this definition, $F_t(\mathbf{X}_t)$ *does not* depend on the scaling of M. Also when the physical units for categories k are chosen so that base year prices $p_0 = e = (1,\dots,1)'$, they remain unchanged when we normalize in this way.

For the $\bar{H}$ estimated by Hu Hui from empirical data [10] and Tabulated in Table 1, the normalized prices $\bar{\pi}_t = p_t/(p_t'\bar{H}p_t)^{1/2}$ had average values $\bar{e}'\bar{\pi}_t$ for years t from 1961 to 1982 which differed only slightly from 1, see the last column of Table 4, so that the investor would be indifferent as to whether the denominator of (4.12.0) or (4.13.0) were used for numeraire.

We need only show that a utility function does not exist for this special case of the equilibrium problem when prices are normalized by $\bar{\pi}_t = p_t/\bar{e}p_t$ and $F_t(\mathbf{X}_t)$ is defined by (4.13.0). Consider a two period model so that we are maximizing the utility $U(\mathbf{X}) = U_1(\mathbf{X}_1) + \delta \mathbf{U}_2(\mathbf{X}_2)$ subject to the primal constraints (4.1.0) and (4.2.0). Further suppose $\mathbf{X}_1$ has only two components so that $\mathbf{X}_1 = (\mathbf{X}_{11}, \mathbf{X}_{12})$. Let $\mathbf{S}_1 = (\mathbf{S}_{11}, \mathbf{S}_{12})$, $\mathbf{S}_1 - \mathbf{X}_1 = (\mathbf{S}_{11} - \mathbf{X}_{11}, \mathbf{S}_{12} - \mathbf{X}_{12})$. Let $\pi_1 = (\pi_{11}, \pi_{12})$ and $M = [m_{ij}]$ be a 2×2 symmetric positive definite matrix. Let $\mathbf{S}_1 - \mathbf{X}_1 = V = (V_1, V_2)$ where

$$V_1 = \mathbf{S}_{11} - \mathbf{X}_{11}, \quad V_2 = \mathbf{S}_{12} - \mathbf{X}_{12}. \tag{4.14.0}$$

At a maximum the Kuhn-Tucker conditions $\partial \mathbf{U}_1/\partial \mathbf{X}_1 = \pi_1$ should hold, see [12]. From (4.13.0), $\pi = MV/D$ where the denominator $D = \bar{e}'MV = \frac{1}{2}(m_{11} + m_{12})V_1 + \frac{1}{2}(m_{12} + m_{22})V_2$. Therefore, the following should hold:

$$\partial \mathbf{U}_1/\partial \mathbf{X}_{11} = \pi_{11} = (m_{11}V_1 + m_{12}V_2)/D, \tag{4.15.1}$$

$$\partial \mathbf{U}_1/\partial \mathbf{X}_{12} = \pi_{12} = (m_{12}V_1 + m_{22}V_2)/D. \tag{4.15.2}$$

In order for a utility function to exist, the second partial

$$\partial^2 \mathbf{U}_1/\partial \mathbf{X}_{11}\partial \mathbf{X}_{12} = -\partial^2 \mathbf{U}_1/\partial \mathbf{X}_{11}\partial V_2$$

computed from (4.15.1) should agree with

$$\partial^2 U_1/\partial \mathbf{X}_{12}\partial \mathbf{X}_{11} = -\partial^2 U_1/\partial \mathbf{X}_{12}\partial V_1$$

computed from (4.15.2) *for all choices of* $\mathbf{X}_{11}, \mathbf{X}_{12}$. Setting these second partials equal to each other, we should have

$$-\frac{m_{12}}{D} + \frac{(m_{11}V_1 + m_{12}V_2)(m_{12} + m_{22})(1/2)}{D^2}$$
$$\equiv -\frac{m_{12}}{D} + \frac{(m_{12}V_1 + m_{22}V_1)(m_{11} + m_{12})(1/2)}{D^2} \tag{4.15.3}$$

which reduces to $(m_{11}m_{22} - m_{12}^2)(V_1 - V_2)/D^2 \equiv 0$. But this does not hold for all choices of $(\mathbf{X}_{11}, \mathbf{X}_{12})$ because M being positive definite implies $m_{11}m_{12} - m_{12}^2 > 0$ and $(V_1 - V_2) = (\mathbf{S}_{11} - \mathbf{X}_{11}) - (\mathbf{S}_{12} - \mathbf{X}_{12}) \neq 0$ for general, $\mathbf{X}$, a contradiction. ∎

Our proof of non-existence of an objective function when prices are normalized in the conventional way is only valid for two or more periods. In a one period case, the factor $\bar{e}'p$ normalizing prices in the rate of return formula can be factored out and replaced by $(p'\bar{H}p)^{1/2}$. The new problem is then equivalent to the original problem but in its new form a utility function exists. However, in the multiperiod case, it is not possible to factor out $\bar{e}'p_t$ and replace it by $(p_t'\bar{H}p_t)^{1/2}$ without invalidating the rate-of-return relation (4.3.0).

Comment.

This ends the theoretical derivation of an objective function for the economy. We conclude that the economy will grow if it has the resources and technology to grow and if it pays according to the aggregate "utility" function to trade off movement of the consumption vector $\mathbf{X}_t$ towards the "satiation" vector $\mathbf{S}_t$ of earlier periods t for considerably larger movements towards the satiation vectors of later periods.

Recalling $v = \bar{S}_t - \bar{X}_t$, $\mathbf{X}_t = P_t \cdot \bar{X}_t$, $\mathbf{S}_t = P_t \cdot \bar{S}_t$ and $\mathbf{I}_t = P_t \bar{I}_t$ where P_t is population size, we have by Euler's Theorem for a *general* homogenous utility function of degree 1:

$$-\mathbf{U}_t = P_t \cdot Z = P_t \cdot (\partial Z/\partial v')v \tag{4.16.0}$$
$$= P_t \cdot \bar{\pi}'v = P_t \cdot (\bar{\pi}'\bar{S}_t - \bar{I}_t) \tag{4.16.1}$$
$$= \bar{\pi}_t'\mathbf{S}_t - \mathbf{I}_t \,. \tag{4.16.2}$$

Thus the disutility is the sum of the discounted *additional aggregate income* (measured in normalized period t prices) needed to purchase the "satiation" vector over various periods t where the less additional income required the higher the "standard of living".

Ideally normalized prices $\bar{\pi}_t$ should have the property that a unit amount of income should enable each individual j to purchase at prices $\bar{\pi}_t$ goods X^j whose utility to j is unity. In a certain geometric mean sense this is true if $\bar{\pi}_t$ are prices p_t normalized by the numeraire $GeometricMean_i(p_t H^i p_t)^{1/2}$. For a representative set of individuals i with utility functions $U_t^i(S^i, H^i)$ for (S^i, H^i) in the urn and income I^i, let $-\hat{U}_t^i = [(S^i - X^i)'M^i(S^i - X^i)]^{1/2}$ be the negative utility of optimal X^i to i. It is easy to prove

$$-\hat{U}_t^i = J^i(\bar{\pi}_t' H^i \bar{\pi}_t)^{-1/2} \tag{4.17.0}$$

where $J^i = \bar{\pi}_t' S^i - I^i$ is the additional income i requires to reach satiation. Letting $\mathcal{G}_i$ denote *geoemtric mean*, it follows that the $\mathcal{G}_i(-\hat{U}_t^i)/\mathcal{G}_i(\hat{J}^i)$ is unity since $\mathcal{G}_i(\bar{\pi}_t' H^i \bar{\pi}_t)^{1/2}$ is unity when $\mathcal{G}_i(p_t H^i p_t)^{1/2}$ is used as numeraire.

PART V:
SIMPLIFYING AND ESTIMATING THE DEMAND AND UTILITY FUNCTIONS

Simplifying the Form of the Per Capita Demand and Utility Functions.

Our immediate goal is to replace the key factors $G(p) = \mathcal{E}_i[(p'H^ip)^{-1} \cdot H^ip]$ associated with the demand function and $-2\log Z = \mathcal{E}_i \log(p'H^ip)$ associated with the utility function by simpler expressions whose parameters are easier to estimate and then give reasons why these approximations may be very good. We will use the symbol $\doteq$ to denote *approximately equal.*

Note that the demand function (1.6.0) of individual i is invariant to the scaling of H^i. Positive definite matrices $H^i = [H^i_{k\ell}]$ satisfy $e'H^ie = \sum_k \sum_\ell H^i_{k\ell} > 0$ where $e' = (1, \ldots, 1)$. Therefore, see (1.1.1), we have assumed without loss of generality that H^i has been rescaled so that

$$e'H^ie = \sum_k \sum_\ell H^i_{k\ell} = 1 \qquad \text{for all } i, \quad e' = (1, \ldots, 1). \tag{5.1.0}$$

The units for measuring the consumption of goods in "physical" terms are usually defined so that their base year prices per unit is $p_0 = e = (1, \ldots, 1)'$. Therefore $p_0'H^ip_0 = 1$ for all i by (5.1.0), implying $p_0'\bar{H}p_0 = 1$ where we define $\bar{H} = \mathcal{E}_iH^i$. It is convenient for the simplification that we use as numeraire $(p'\bar{H}p)^{1/2}$ implying $\bar{p}'\bar{H}\bar{p} = 1$ for all "normalized" vectors $\bar{p}$. It follows, in particular, that $\bar{p}_0 = p_0/(p_0'\bar{H}p_0)^{1/2} = p_0 = e$ and $\bar{p}_0'H^i\bar{p}_0 = 1$ for all i for the base year.

Theorem 5.1. *The first-order approximation of* $\log Z$ *as a function of* $p = G^{-1}(\bar{S}_t - \bar{X}_t)$ *yields the approximation:*

$$Z = \mathcal{G}_i\ (pH^ip)^{-1/2} \doteq (p'\bar{H}p)^{-1/2}\ , \tag{5.2.0}$$

where positive definite H^i *are rescaled so that* $e'H^ie = 1$ *for all* i *and* $\bar{H} = \mathcal{E}_iH^i$ *and* $\mathcal{G}_i$ *denotes the geometric mean.*

Proof. We normalize p by $\bar{p} = p/(p'\bar{H}p)^{1/2}$. Subtracting $\log(p\bar{H}p)$ from both sides of $-2\log(Z)$, and letting $\bar{p} = e + \Delta$, we obtain

$$\begin{aligned}
-2\log(Z) - \log(p'\bar{H}p) &= \mathcal{E}_i \log[(p'H^ip)/(p'\bar{H}p)] = \mathcal{E}_i \log(\bar{p}'H^i\bar{p}) && (5.3.1)\\
&= \mathcal{E}_i \log[(e+\Delta)'H^i(e+\Delta)] && (5.3.2)\\
&= \mathcal{E}_i \log(e'H^ie + 2\Delta'H^ie + \Delta'H^i\Delta) && (5.3.3)\\
&= \mathcal{E}_i \log(1 + 2\Delta'H^ie + \Delta'H^i\Delta). && (5.3.4)
\end{aligned}$$

As a *first order approximation* of a natural log, set $\log(1 + \epsilon_i) \doteq \epsilon_i$.

$$\begin{aligned}
-2\log(Z) - \log(p'\bar{H}p) &\doteq \mathcal{E}_i[(2\Delta'H^ie) + (\Delta'H^i\Delta)] = 2\Delta'\bar{H}e + \Delta\bar{H}\Delta && (5.4.1)\\
&\doteq (1 + 2\Delta'\bar{H}e + \Delta\bar{H}\Delta) - 1 = (e+\Delta)'\bar{H}(e+\Delta) - 1 && (5.4.2)\\
&\doteq \bar{p}'\bar{H}\bar{p} - 1 = 1 - 1 = 0\ , && (5.4.3)
\end{aligned}$$

where $\bar{p}'\bar{H}\bar{p} = 1$ was shown in comments following (5.1.0). Therefore, the approximation of $Z(p)$ is given by (5.2.0). ∎

Little, of course, is known about the utility functions of individuals or how they vary about $\bar{H}$. For the experiment we are about to describe, our particular choice of distribution for $h_i = \bar{p}'H^i\bar{p}$, is designed to show that the approximations of $Z(p)$ and hence $G(p) = -\partial(\log Z)/\partial p$ are good ones for variable $\bar{p}$ in the neighborhood of some fixed $\bar{p}_0 = (1, 1, \ldots,)' = e$ even if the H^i for individuals i were to vary a lot. In Table 4, prices p_t for the years $t = 1961$ to 1982 relative to base year 1972 prices $p_0 = e$, are tabulated. For $\bar{p}$ *in the neighborhood* of $\bar{p}_0 = e$, the values of $h_i = \bar{p}'H^i\bar{p}$ will obviously be distributed around the mean $\bar{p}'\bar{H}\bar{p} = 1$ with a *very* low standard deviation, because $\bar{p}_0'H^i\bar{p}_0 = 1$ for all i. The vector of prices for 1982 differed the most from the base year with some prices differing by more than 30%. Therefore p_t for $t = 1982$ was selected as an extreme case of p_t for our illustrative example.

The following experiment was then made. The symmetric matrix, denoted $\bar{H}$ in Table 1, was inverted to produce an M and a thousand cases of random M^i were generated by independently varying each symmetric pair $M^i_{k\ell}$ about $M_{k\ell}$ by $(1+\theta)M_{k\ell}$ where θ was binomially distributed with mean 0, and standard deviation .1, and maximum range $-.2 \le \theta \le .2$. Three of the thousand cases were dropped because M^i turned out not to be positive definite. The inverses $[M^i]^{-1} = H^i$ were then computed and H^i replaced by rescaled H^i so that $e'H^ie = 1$. We will now call $\bar{H}$, the average of these H^i and denote by $\bar{p} = p_t/(p_t'\bar{H}p_t)^{1/2}$ where p_t is the price vector for year 1982. This makes the mean of $\bar{p}H^i\bar{p} = 1$. The standard deviation of $\bar{p}H^i\bar{p}$ for the sample of 997 cases turned out to be less that .05 or 5%. For our analysis, we exaggerated the standard deviation of $h_i = \bar{p}'H^i\bar{p}$ to be 10% instead of 5% and to have a maximum range of 20% about their mean of 1. If 5% instead of 10% were used in Theorem 5.2 below, the error ϵ would have been negligible.

Theorem 5.2. *Let $\bar{p} = p_t/(p_t'\bar{H}p_t)$. If the distribution of $h_i = (\bar{p}'H^i\bar{p}) > 0$ is binomial with mean of 1, standard deviation $= .1$, and a range $.8 \le \bar{p}'H^i\bar{p} \le 1.2$, then the percent error in the approximation of Z by $(p_t'\bar{H}p_t)^{-1/2}$ is less than 0.5%:*

$$-2\log(Z) = \log[(1+\epsilon)(p'\bar{H}p)] \quad \textit{where } |\epsilon| < 0.5\% \,. \tag{5.5.0}$$

Proof. We seek an error bound for ϵ. From (5.5.0) and (5.2.0),

$$\log[(1+\epsilon)(p_t'\bar{H}p_t)] = \mathcal{E}_i[\log(p_t'H^ip_t)] \,, \tag{5.6.1}$$

$$\log[(1+\epsilon)] = \mathcal{E}_i[\log(\bar{p}'H^i\bar{p})] \,, \qquad \bar{p} = p/(p'\bar{H}p)^{1/2} \,, \tag{5.6.2}$$

Our task is to prove $|\epsilon| \le .005$. We assumed that $h_i = \bar{p}'H^i\bar{p}$, roughly speaking, has a truncated normal distribution with mean of 1, a standard deviation less than .1, and $.8 \le h_i \le 1.2$, namely we assumed it to be the binomial distribution with values (.8, .9, 1.0, 1.1, 1.2) and corresponding probabilities $(1/16, 4/16, 6/16, 4/16, 1/16)$. For this experiment, we therefore have:

$$\log(1+\epsilon) = \frac{1}{16}\log .8 + \frac{4}{16}\log .9 + \frac{6}{16}\log 1.0 + \frac{4}{16}\log 1.1 + \frac{1}{16}\log 1.2 \,, \tag{5.7.0}$$

with an error $|\epsilon| = .00505$. We assert without proof that any continuous unimodal distribution with same mean, standard deviation, and truncation would yield about the same error $|\epsilon| \leq .005$. ∎

Because $\log(Z)$ where Z is the disutility function is so closely approximated by $-\frac{1}{2}\log(p_t'\bar{H}p_t)$ for reasonable variability of $p_t'H^ip_t$, we apply the approximation to $-\partial(\log Z)/\partial p$ to obtain our approximation for the key factor $G(p)$ of the demand function:

$$G(p) = \mathcal{E}_i(p'H^ip)^{-1} \cdot H^ip = -\partial(\log Z)/\partial p \tag{5.8.1}$$

$$\doteq \tfrac{1}{2}\partial[\log(p'\bar{H}p)]/\partial p = (p'\bar{H}p)^{-1} \cdot \bar{H}p \qquad \text{for all } p \text{ near } p_0 \,. \tag{5.8.2}$$

Hence the approximations:

$$v \doteq (p'\bar{H}p)^{-1} \cdot \bar{H}p \,, \qquad p \doteq (v'Mv)^{-1} \cdot Mv \,, \tag{5.8.3}$$

where $M = \bar{H}^{-1}$ and $v = \bar{S}_t - \bar{X}_t$. We now substitute these approximations into various demand function theorems and summarize them here:

Demand functions of individuals with income I (Theorem 1.1):

$$S^j - X^j = (p'S^j - I)(p'H^jp)^{-1} \cdot H^jp \tag{5.9.0}$$

where $I_j^* \leq I \leq I_j^{**} = pS^j$.

The approximation for Theorem 1.3, the expected demand function of individuals whose income is I:

$$\bar{S} - \bar{X}^I = (p'\bar{S} - I)\mathcal{E}_i(p'H^ip)^{-1} \cdot H^ip \doteq (p'\bar{S} - I)(p'\bar{H}p)^{-1} \cdot \bar{H}p \tag{5.10.0}$$

where $\max I_i^* = I^* \leq I \leq I^{**} = \min p'S^i$, H^i rescaled so that $e'H^ie = 1$ and $\bar{H} = \mathcal{E}_iH^i$.

The approximation for Theorem 1.5, the expected per capita demand function:

$$\bar{S} - \bar{X} = (p'\bar{S} - \bar{I})\mathcal{E}_i(p'H^ip)^{-1} \cdot H^ip \doteq (p'\bar{S} - \bar{I})(p'\bar{H}p)^{-1} \cdot \bar{H}p \tag{5.11.0}$$

where $\bar{I}^* \leq \bar{I} \leq \bar{I}^{**}$.

The approximation for Theorem 2.9, the utility function $\bar{U}_t = \bar{U}_t(\bar{X}_t)$ for period t which implies the per capita demand function (5.11.0) is:

$$\bar{U}_t = -\mathcal{G}_i(p'H^ip)^{-1/2} \tag{5.12.1}$$

$$= -(p'\bar{H}p)^{-1/2} = -(v'Mv)^{1/2} \tag{5.12.2}$$

$$\doteq -[(\bar{S} - \bar{X}_t)'M(\bar{S} - \bar{X}_t)]^{1/2}, \tag{5.12.3}$$

where $M = \bar{H}^{-1}$ and $\mathcal{G}_i$ denotes the geometric mean for H^i in the population urn. We now apply the approximation (5.12.3) to Theorem 3.2, after rescaling for population size.

Theorem 5.3. *The equilibrium problem is approximately equivalent to solving the concave program:*

$$\max \mathbf{U}(\mathbf{X}) \doteq -\sum_t \delta^{t-1}[(\mathbf{S}_t - \mathbf{X}_t)' M_t (\mathbf{S}_t - \mathbf{X}_t)]^{1/2}, \tag{5.13.0}$$

subject to $(\mathbf{X}_t, \mathbf{Y}_t) \geq 0$ *and the primal flow conditions* $t = 1, \ldots, T$*:*

$$B_t \mathbf{Y}_t \leq D_{t-1}\mathbf{Y}_{t-1} + k_t \tag{5.13.1}$$

$$-A_t \mathbf{Y}_t + \mathbf{X}_t \leq f_t \tag{5.13.2}$$

where $\mathbf{S}_t, \mathbf{X}_t, \mathbf{Y}_t$ *are aggregate quantities, and* $M_t = \bar{H}_t^{-1}$*,* $\bar{H}_t = \mathcal{E}_i H_t^i$*, and* H_t^i *have been rescaled so that* $\sum_k \sum_\ell H_t^i(k, \ell) = 1$.

EMPIRICAL EVIDENCE OF LINEARITY OF THE DEMAND FUNCTION AT FIXED PRICES

M. Avriel in his studies [2] convoluted the income distribution with average observed personal consumption data for over fifty commodities as a function of average observed income per person *at various household income levels.* More recently, one of the authors, McAllister, repeated the same experiment with more recent survey data for certain key commodities. For their studies a distribution of income for future periods was assumed to be a certain known function of attained per capita income, namely (a) that $C_k(I)$, the average of consumption of individuals at income I at fixed prices p of commodity k will not change in the future; (b) that per capita income $\bar{I}$ may increase with time; and (c) that the distribution of income about $\bar{I}$ *will retain its same shape* when rescaled proportional to $\bar{I}$. The base year distribution was based on survey data. They then determined per capita consumption as a function of per capita income by the convolution formulas (1.10.1) and (1.10.2). The resulting per capita demand functions *turned out to be remarkably linear at fixed prices* over a wide range of per capita income. See references [9, 2, 21].

Tables 2A and 2B tabulate for survey years 1972-73 and 1980-81 the average consumption of U.S. per person in various household income classes of *Food, Clothing, Housing, Housing Operations, Transportation, Recreation, Personal Care,* and *All Other.* Each of these 8 categories of consumption k was plotted against the average income used for consumption per person in each household income class. See Figures 1 to 8 immediately following Tables 2A and 2B. Before plotting, however, the average consumption per person in dollars in each category was divided by its survey year price thereby converting the units for measuring the amount of consumption to "physical" units. In order to exhibit the plots for the two survey years for each category and make them on the same graph comparable, the average income per person in each household income class for the 1980-81 survey in 1980$ was deflated to 1972$. Prices used for converting the consumption in survey year dollars to "physical" units can be found in Table 4. The units for each category k are chosen so that price in 1972-73 for each k is unity. Prices for other years are deflated to 1972$.

Since the vector of prices in each survey year is fixed, the regression of average consumption of category k per person in a household income class versus average income per person should be

linear for "the range of income of interest" according to Theorems 1.2 and 1.3 or its simplified form for estimating its parameters, (5.10.0). The reader is encouraged to ignore the trend lines on Figures 1 to 8 (which will be explained later) and to study the sixteen sets of plotted points and judge for himself how linear they are for these very broad categories of consumption.

Tables 3 and 4 tabulate per capita consumption and prices by category in the years 1961 - 1982. We found the family household data, Tables 2A and 2B, somewhat inconsistent with per capita data of Table 3 possibly because they came from different types of surveys. It seemed best to use the graphs of Tables 2A and 2B only for the purpose of extrapolating a guess of $\bar{S}$. Using this guess, the data of Tables 3 and 4 were then used to estimate a positive definite $\bar{H}$.

Estimating $\bar{S}$. According to Theorems 1.2 or 1.3, the expected demand function of persons at various income levels is linear in income up to a level that some individuals can buy their satiation vector. If we assume for the moment that this is true empirically and the income level just sufficient to buy the expected satiation vector $p\bar{S}$ is known (and not much different for different individuals i), then the satiation value $\bar{S}(k)$ for the k-th category could be found by reading off the ordinate value when the abscissa is equal to $p\bar{S}$. Since the price vectors for the two survey years are slightly different their expected satiation income $p\bar{S}$ in 1972\$ could be different, but it is not too unreasonable to assume for purposes of roughly estimating $\bar{S}$ that the two expected satiation incomes are equal.

With this rational in mind, straight lines were fitted to the data for survey years 1972-73 and 1980-81. The two straight lines shown on Figures 1 to 8 for each category k are "eyeball" fits to the data with greater weight given to the high end. It was assumed that at some very high income level the ordinates of the two fitted lines for each category k would be sufficiently close to one another that making the two ordinate values equal would distort very little the fits to the observed data at the much lower income levels. We arbitrarily pegged this high per capita income level to be \$25,510, an income level about equivalent to that of a three person household income of \$200,000 in 1987 dollars. Therefore, the abscissa where the two lines intersect is at \$25,510. Their common ordinate value is the estimate of $\bar{S}(k)$. This was done for each k except for $k = Recreation$ whose trend line in 1972-73 was ignored because it had a radically higher slope from that of 1980-81. On Figure 8 only one trend line is shown; this means that two trend lines were estimated to be the same. The estimates for $\bar{S}(k)$ are tabulated in Table 1.

Admittedly this is a pretty crude "eye ball" way to estimate $\bar{S}$, nevertheless the resulting linear fits appear to be reasonably good in most cases. The real test, however, is not how good is the estimate of $\bar{S}$ but how good is the per capita demand function (5.11.0), found using this estimate of $\bar{S}$ and price and per capita consumption data in various years to estimate $\bar{H}$.

Estimating the positive definite matrix $\bar{H} = M^{-1}$.

The values estimated for the elements of matrix $\bar{H}$ are also tabulated in Table 1. The matrix

$H = [\bar{H}_{ij}]$ was estimated by a technique of solving a semi-infinite linear program recently developed by Hu Hui [10] that finds a least square fit under the restrictions that $\bar{H}$ be symmetric, positive definite and $\Sigma\Sigma\bar{H}_{ij} = 1$. The least square fit was made to (5.11.0) after multiplying the equation by $p'\bar{H}p$. The source data for this estimate was (a) the guess of satiation levels $\bar{S}$ described above, and (b) the *observed* $\bar{X}_t$, which we denote by $\tilde{X}_t$, and *observed* prices p_t from the national income and product accounts of the U.S. for years t from 1961 to 1982. Per capita income was assumed to be the observed values $\bar{I}_t = p_t'\tilde{X}_t$. *Observed* $\tilde{X}_t$, which are indices of physical quantities of consumption, are tabulated in 1972$ in Table 3 for each of the eight categories for each year t from 1961 to 1982. The corresponding prices p_t for these items deflated to base year 1972 prices are tabulated in Table 4. The predicted values of $\bar{X}_t$, using the per capita demand function (5.11.0) with the estimated parameter values given in Table 1, are tabulated in Table 5 for each of the 22 years. A comparison of how well predicted $\bar{X}_t$ compares with observed $\tilde{X}_t$ can be made by comparing Table 3 with Table 5 or by comparing the solid line curves with the broken line curves in Figures 9 to 16 immediately following Table 5. Note the excellence of the fit. The average percent error of fit can be found at the bottom of Table 5.

The *average* price, after the price vector p_t is normalized by using $(p_t'\bar{H}p_t)^{1/2}$ as numeraire, are tabulated in the last column of Table 4. It can be seen that these differ only slightly from unity, which would have been their average, had they been normalized using average price as numeraire. Thus an investor would be quite indifferent as to which of the two ways is chosen to normalize future prices before calculating a rate of return.

Table 1

ESTIMATED DEMAND FUNCTION: $\bar{S} - \bar{X}_t = \dfrac{p_t'\bar{S} - \bar{I}_t}{p_t'\bar{H}p_t} \cdot \bar{H}p_t$

Entries in columns below are estimated $\bar{H}_{k\ell} \times 1000$ for $k \geq \ell$

k	Est.	Food $\ell = (1)$	Cloth (2)	Housing (3)	House Oper. (4)	Transp. (5)	Recreation (6)	Per. Care (7)	Other (8)
(1) Food	2666	**30**							
(2) Clothing	1970	1	**35**						
(3) Housing	4670	7	33	**77**					
(4) House Oper.	2050	− 6	−14	20	**110**				
(5) Transp.	3479	10	9	8	54	**54**			
(6) Recreation	2253	18	−17	− 2	− 3	20	**56**		
(7) Per. Care	526	12	20	− 7	−77	− 1	− 1	**97**	
(8) Other	7340	16	15	54	49	32	23	−18	**145**

Entries $\bar{H}_{k\ell}$ above the diagonal $= \bar{H}_{\ell k}$ below the diagonal

p_t = observed price vector (period t) from time series (Table 3)

$\tilde{X}_t$ = observed per capita consumption vector from time series (Table 4)

$\bar{I}_t$ = observed per capita total consumption income $= p_t'\tilde{X}_t$ (Total Column, Table 4)

$\bar{S}$ = estimated per capita satiation vector

$\bar{H}$ = estimated positive definite symmetric matrix*

$\bar{X}_t$ = per capita consumption vector predicted by demand function

* Estimated by a method developed by Hu Hui that yields least square fit subject to $\bar{H}$ being positive definite [10].

Table 2A

TABLE 2A

EXPENDITURES PER PERSON IN 1972-73 (1972$) BY HOUSEHOLD INCOME CLASS

HOUSEHOLD INCOME	FAMILY SIZE	FOOD	CLOTHING	HOUSING	HOUSING OPERATIONS	TRANS-PORTATION	RECREATION	PERSONAL CARE	ALL OTHER	TOTAL EXPENDITURE
UNDER $3000	1.4	635	96	833	128	309	168	64	297	2530
$3,000 - 4,000	1.9	671	109	726	128	302	183	64	324	2507
$4,000 - 5,000	2.1	658	113	702	128	363	184	60	352	2560
$5,000 - 6,000	2.4	623	120	649	134	354	193	62	366	2501
$6,000 - 7,000	2.5	652	127	669	144	450	196	70	431	2739
$7,000 - 8,000	2.7	661	139	714	156	445	216	74	409	2804
$8,000 - 10,000	2.8	675	153	662	160	501	239	68	508	2966
$10,000 - 12,000	3.2	680	160	638	163	536	243	64	528	3012
$12,000 - 15,000	3.4	680	171	668	186	567	273	63	598	3206
$15,000 - 20,000	3.6	742	209	759	223	627	328	74	720	3682
$20,000 - 25,000	3.8	782	249	790	253	698	412	81	881	4146
$25,000 and over	3.8	913	361	1038	383	820	597	112	1406	5630

SOURCE: U.S. Consumer Expenditure Survey, 1972-73; NIPA Categories
OTHER = Medical Care + Personal Business + Private Education + Research + Religious and Charity Contributions

This summary was prepared by Dorothy Sheffield; Figures 1 to 8 that follow display the above graphically. These were prepared by Isabel Pereira.

Table 2B

TABLE 2B

EXPENDITURES PER PERSON IN 1980-81 (1980$) BY HOUSEHOLD INCOME CLASS

HOUSEHOLD INCOME	FAMILY SIZE	FOOD	CLOTHING	HOUSING	HOUSING OPERATIONS	TRANS-PORTATION	RECREA-TION	PERSONAL CARE	ALL OTHER	TOTAL EXPENDITURE
Under 5,000	1.7	932	177	1013	224	639	299	90	696	4070
$5,000 - 10,000	2.1	992	177	1036	327	870	325	93	750	4570
$10,000 - 15,000	2.5	1084	207	1106	292	1105	366	105	883	5148
$15,000 - 20,000	2.7	1228	248	1230	336	1280	423	112	1141	5998
$20,000 - 30,000	3.1	1283	275	1475	388	1425	464	120	1345	6775
$30,000 and over	3.4	1453	435	1997	620	1759	644	186	2205	9299
					DEFLATED TO 1972$					
	DEFLATORS ÷	2.037	1.386	1.998	1.891	2.196	1.519	2.087	2.080	
Under 5000	1.7	458	127	507	118	291	197	43	335	2076
$5,000 - 10,000	2.1	487	127	519	173	396	214	44	360	2321
$10,000 - 15,000	2.5	532	150	554	154	503	241	50	425	2608
$15,000 - 20,000	2.7	602	179	616	178	583	279	54	549	3039
$20,000 - 30,000	3.1	630	198	738	205	649	305	58	646	3430
$30,000 and over	3.4	713	314	999	328	801	424	89	1060	4729

SOURCE: U.S. Consumer Expenditure Survey, 1980-81; NIPA Categories
OTHER = Medical Care + Personal Business + Private Education + Research + Religious and Charity Contributions

This summary was prepared by Dorothy Sheffield; Figures 1 to 8 that follow display the above graphically. These were prepared by Isabel Pereira.

Figure 1

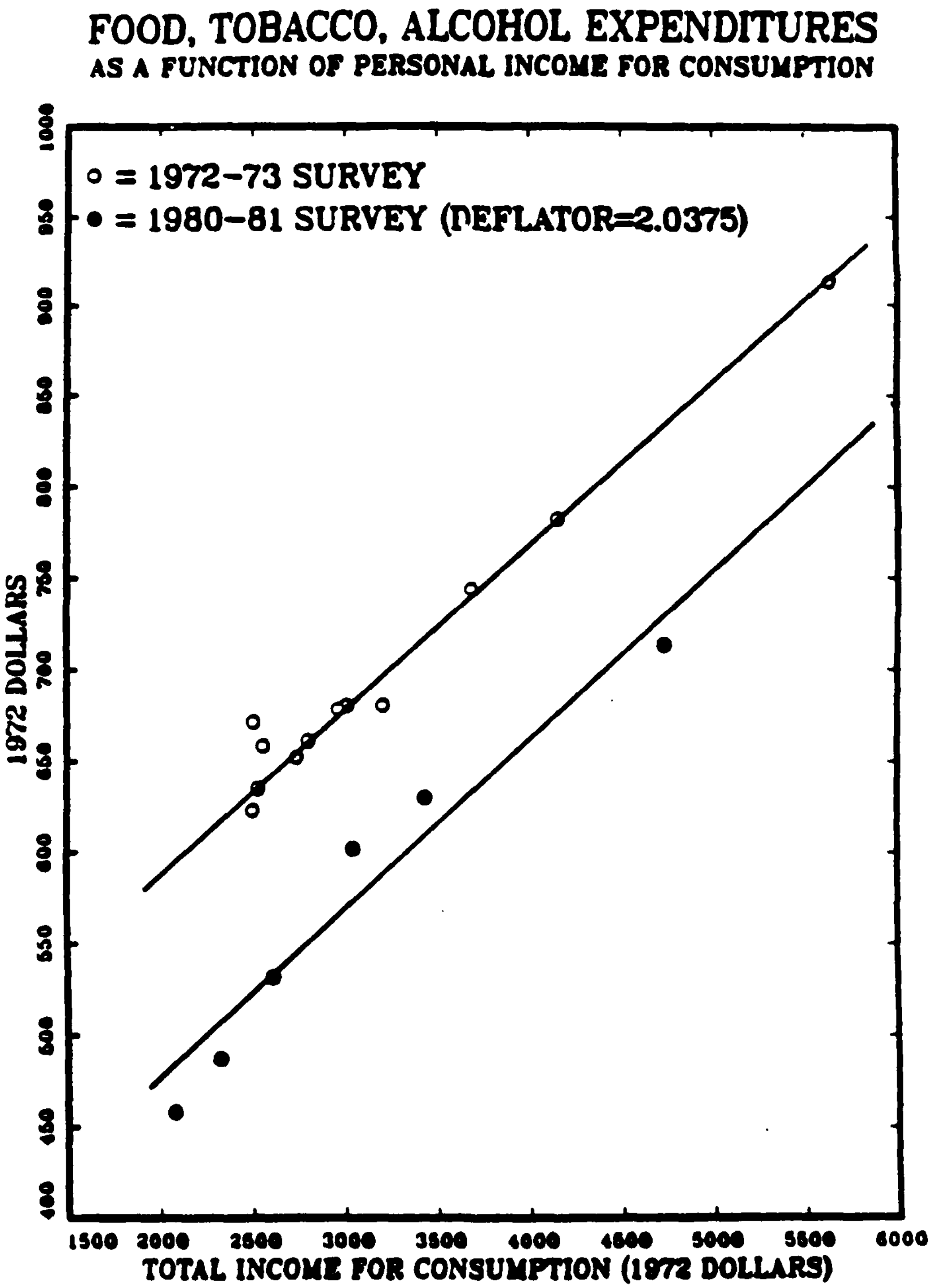

FOOD, TOBACCO, ALCOHOL EXPENDITURES
AS A FUNCTION OF PERSONAL INCOME FOR CONSUMPTION
○ = 1972–73 SURVEY
● = 1980–81 SURVEY (DEFLATOR=2.0375)
1972 DOLLARS
400
450
500
550
600
650
700
750
800
850
900
950
1000
1500
2000
2500
3000
3500
4000
4500
5000
5500
6000
TOTAL INCOME FOR CONSUMPTION (1972 DOLLARS)

Figure 2

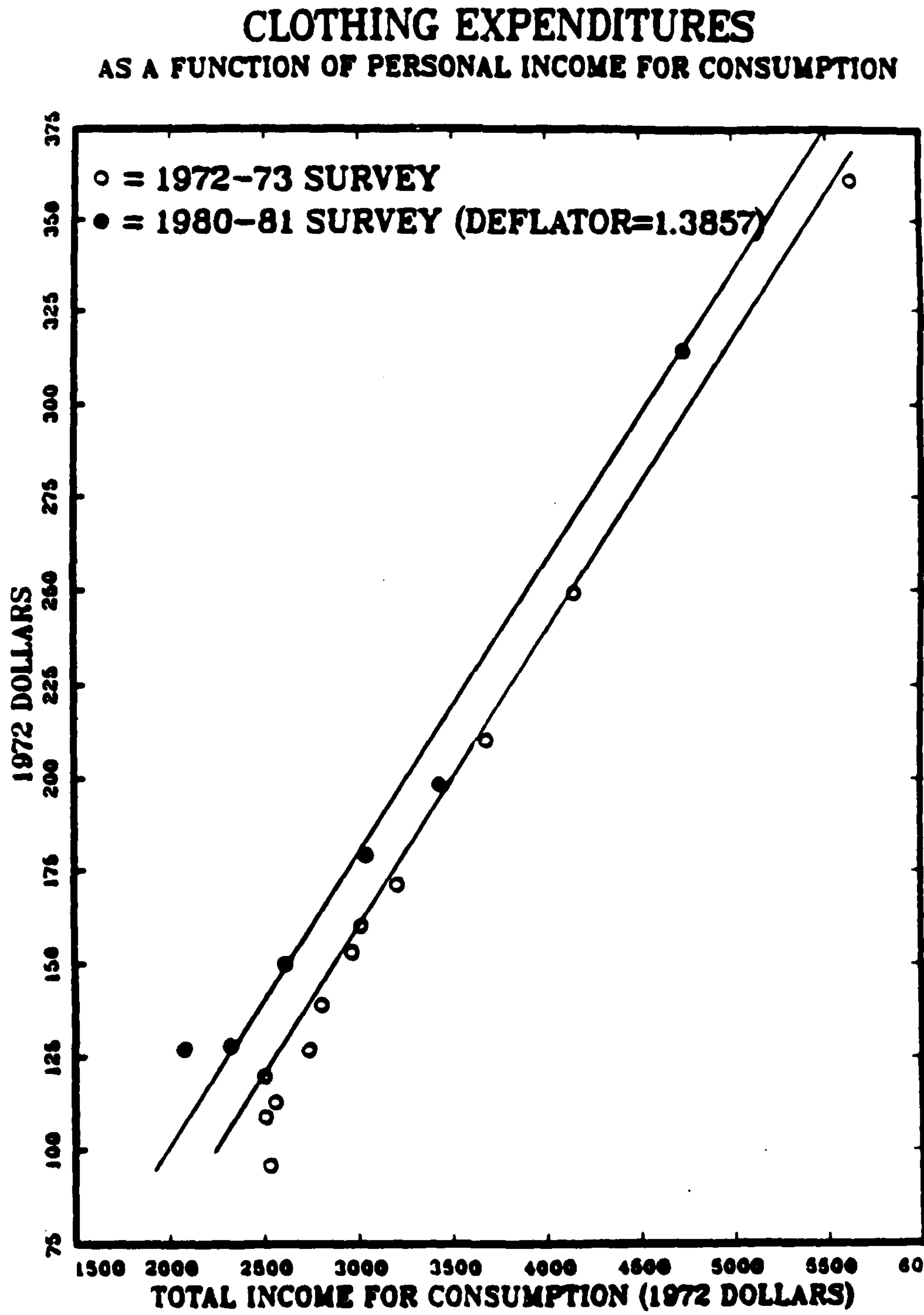

CLOTHING EXPENDITURES
AS A FUNCTION OF PERSONAL INCOME FOR CONSUMPTION
○ = 1972-73 SURVEY
● = 1980-81 SURVEY (DEFLATOR=1.3857)
1972 DOLLARS
75
100
125
150
175
200
225
250
275
300
325
350
375
1500
2000
2500
3000
3500
4000
4500
5000
5500
6000
TOTAL INCOME FOR CONSUMPTION (1972 DOLLARS)

Figure 3

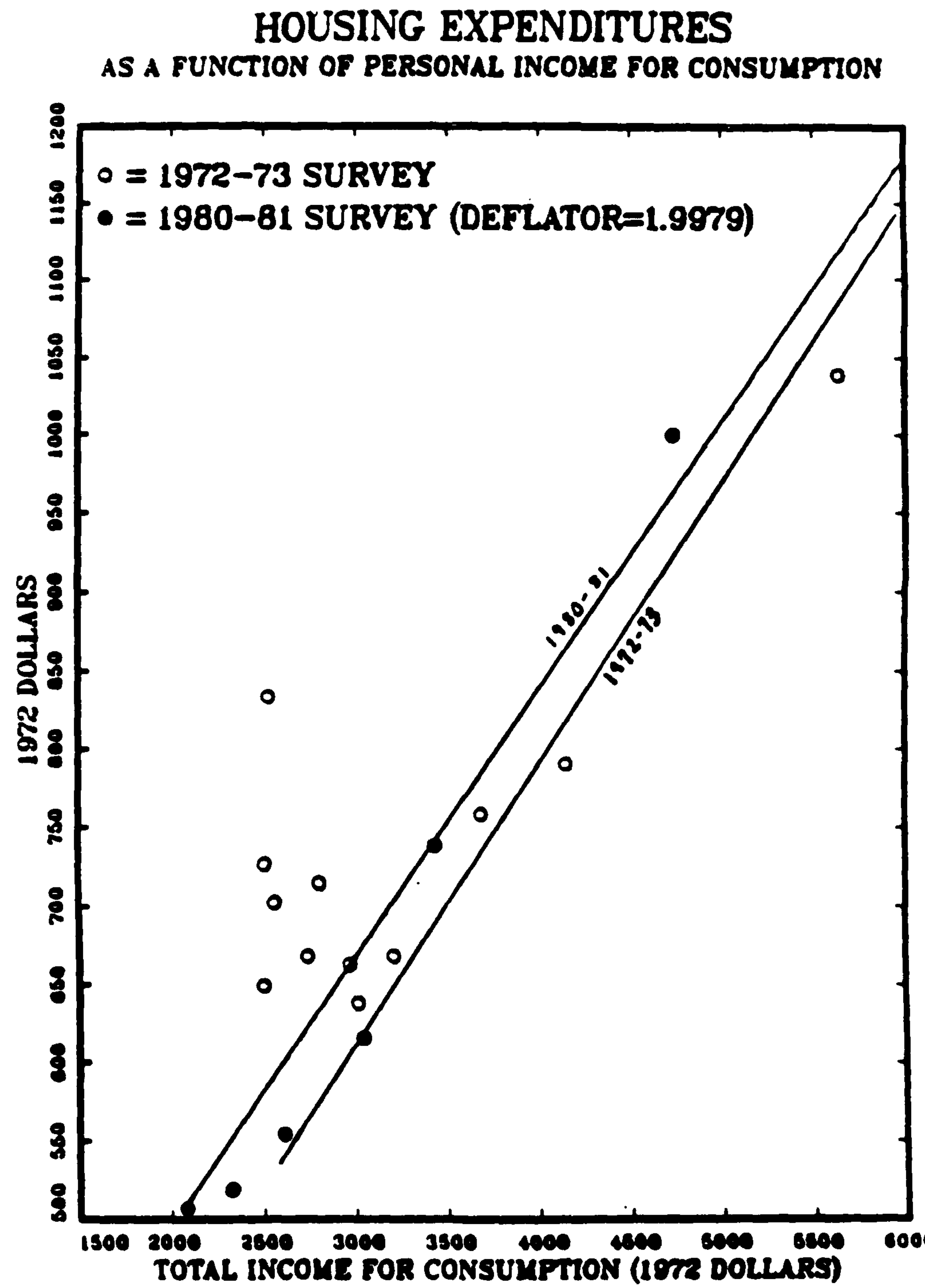

HOUSING EXPENDITURES
AS A FUNCTION OF PERSONAL INCOME FOR CONSUMPTION
○ = 1972-73 SURVEY
● = 1980-81 SURVEY (DEFLATOR=1.9979)
1980-81
1972-73
1972 DOLLARS
TOTAL INCOME FOR CONSUMPTION (1972 DOLLARS)
1500
2000
2500
3000
3500
4000
4500
5000
5500
6000
500
550
600
650
700
750
800
850
900
950
1000
1050
1100
1150
1200

Figure 4

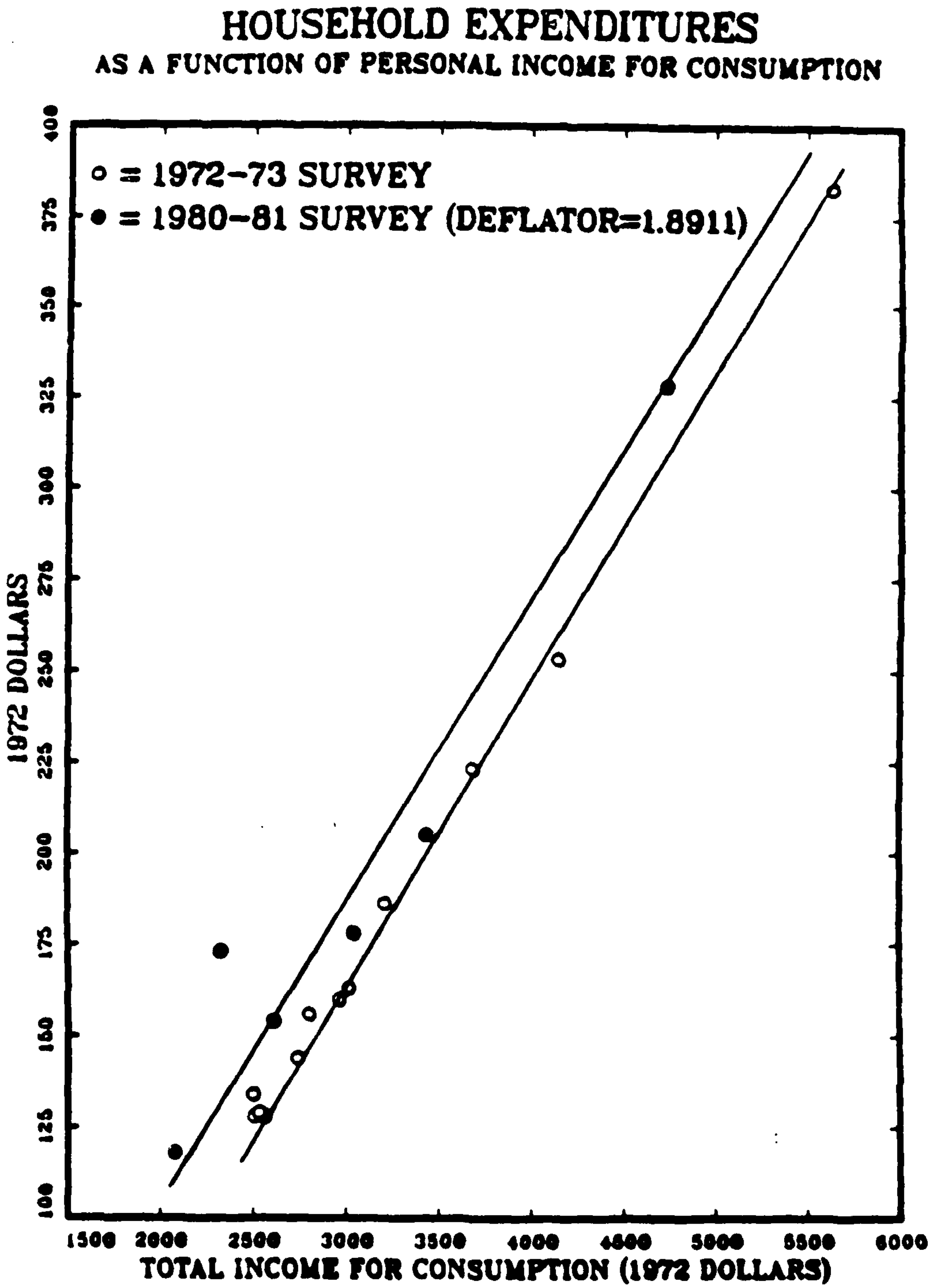
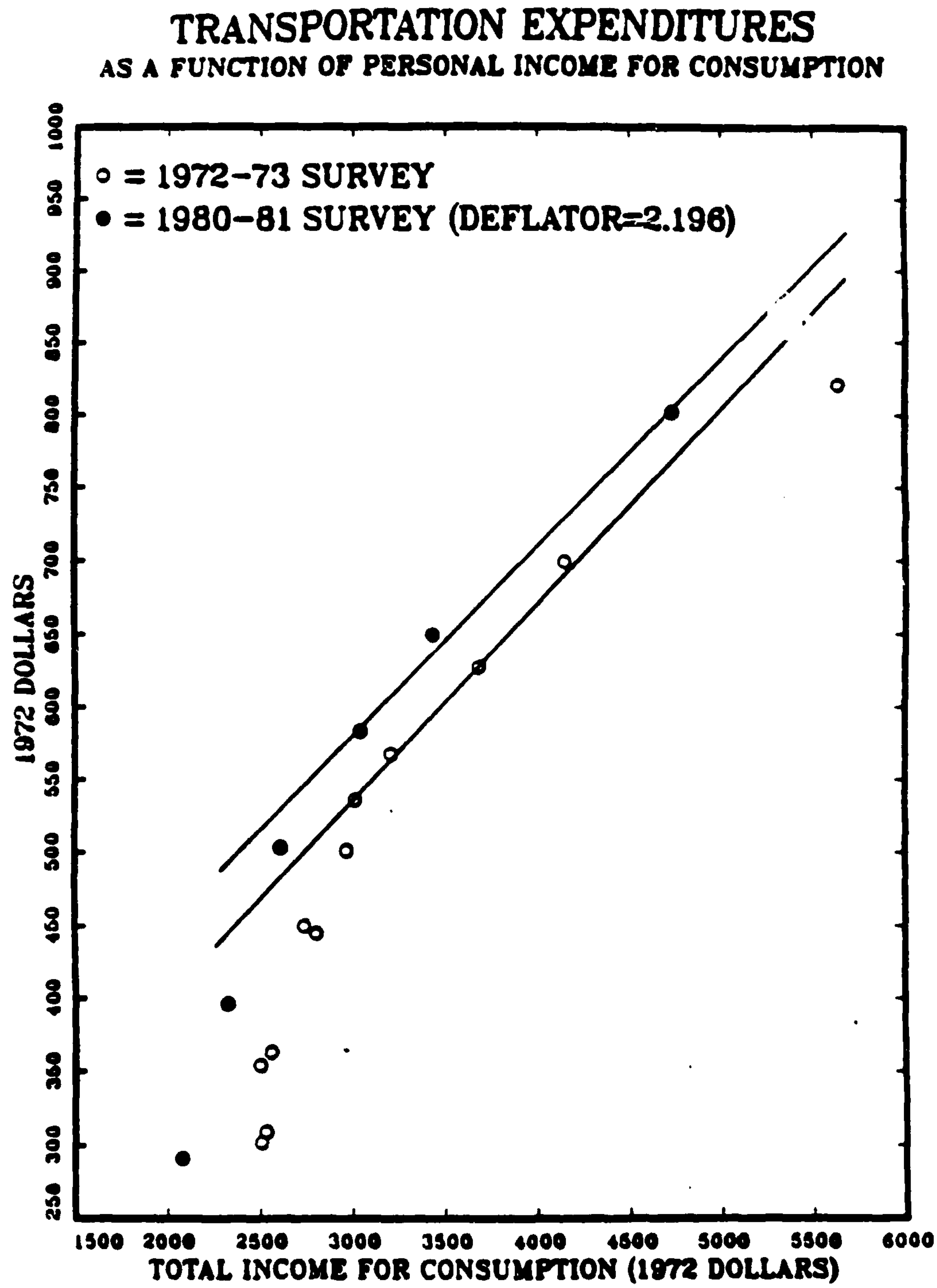

HOUSEHOLD EXPENDITURES
AS A FUNCTION OF PERSONAL INCOME FOR CONSUMPTION
○ = 1972-73 SURVEY
● = 1980-81 SURVEY (DEFLATOR=1.8911)
1972 DOLLARS
100
125
150
175
200
225
250
275
300
325
350
375
400
1500
2000
2500
3000
3500
4000
4500
5000
5500
6000
TOTAL INCOME FOR CONSUMPTION (1972 DOLLARS)

Figure 5



TRANSPORTATION EXPENDITURES
AS A FUNCTION OF PERSONAL INCOME FOR CONSUMPTION
○ = 1972-73 SURVEY
● = 1980-81 SURVEY (DEFLATOR=2.196)
1972 DOLLARS
250
300
350
400
450
500
550
600
650
700
750
800
850
900
950
1000
1500
2000
2500
3000
3500
4000
4500
5000
5500
6000
TOTAL INCOME FOR CONSUMPTION (1972 DOLLARS)

Figure 6

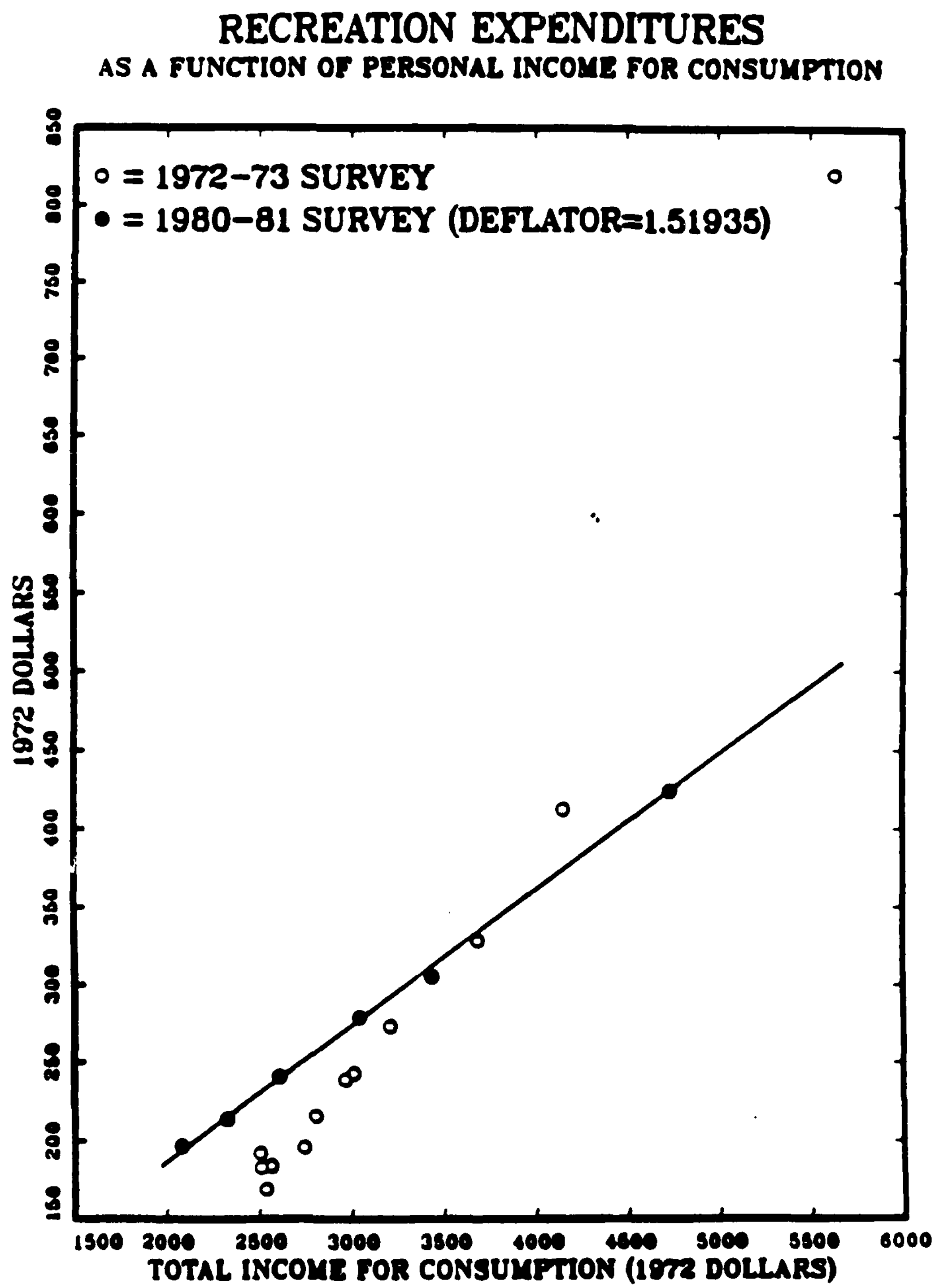

RECREATION EXPENDITURES
AS A FUNCTION OF PERSONAL INCOME FOR CONSUMPTION
○ = 1972-73 SURVEY
● = 1980-81 SURVEY (DEFLATOR=1.51935)
1972 DOLLARS
150
200
250
300
350
400
450
500
550
600
650
700
750
800
850
1500
2000
2500
3000
3500
4000
4500
5000
5500
6000
TOTAL INCOME FOR CONSUMPTION (1972 DOLLARS)

Figure-7

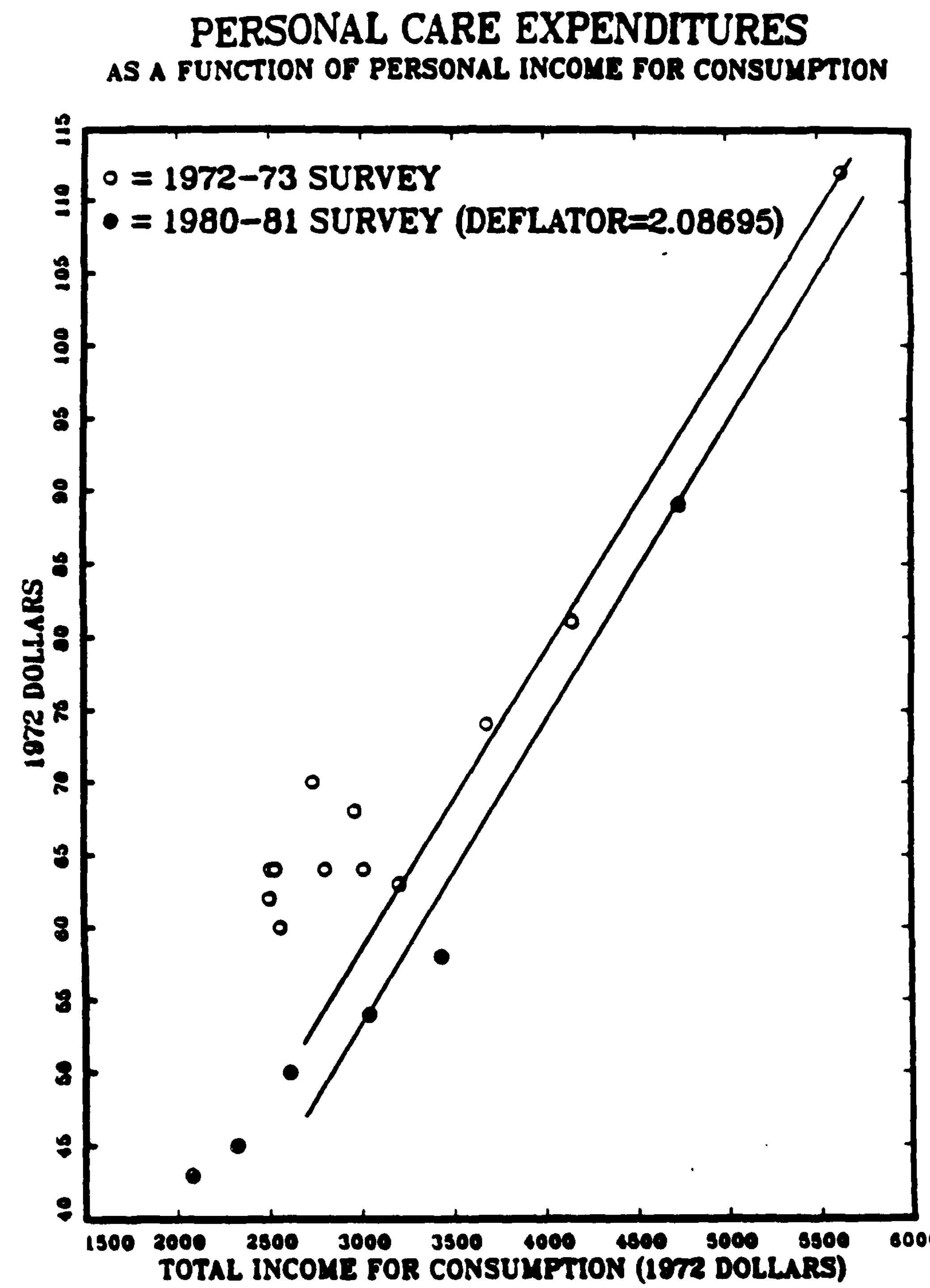

PERSONAL CARE EXPENDITURES
AS A FUNCTION OF PERSONAL INCOME FOR CONSUMPTION
○ = 1972-73 SURVEY
● = 1980-81 SURVEY (DEFLATOR=2.08695)
1972 DOLLARS
40
45
50
55
60
65
70
75
80
85
90
95
100
105
110
115
1500
2000
2500
3000
3500
4000
4500
5000
5500
6000
TOTAL INCOME FOR CONSUMPTION (1972 DOLLARS)

Figure 8

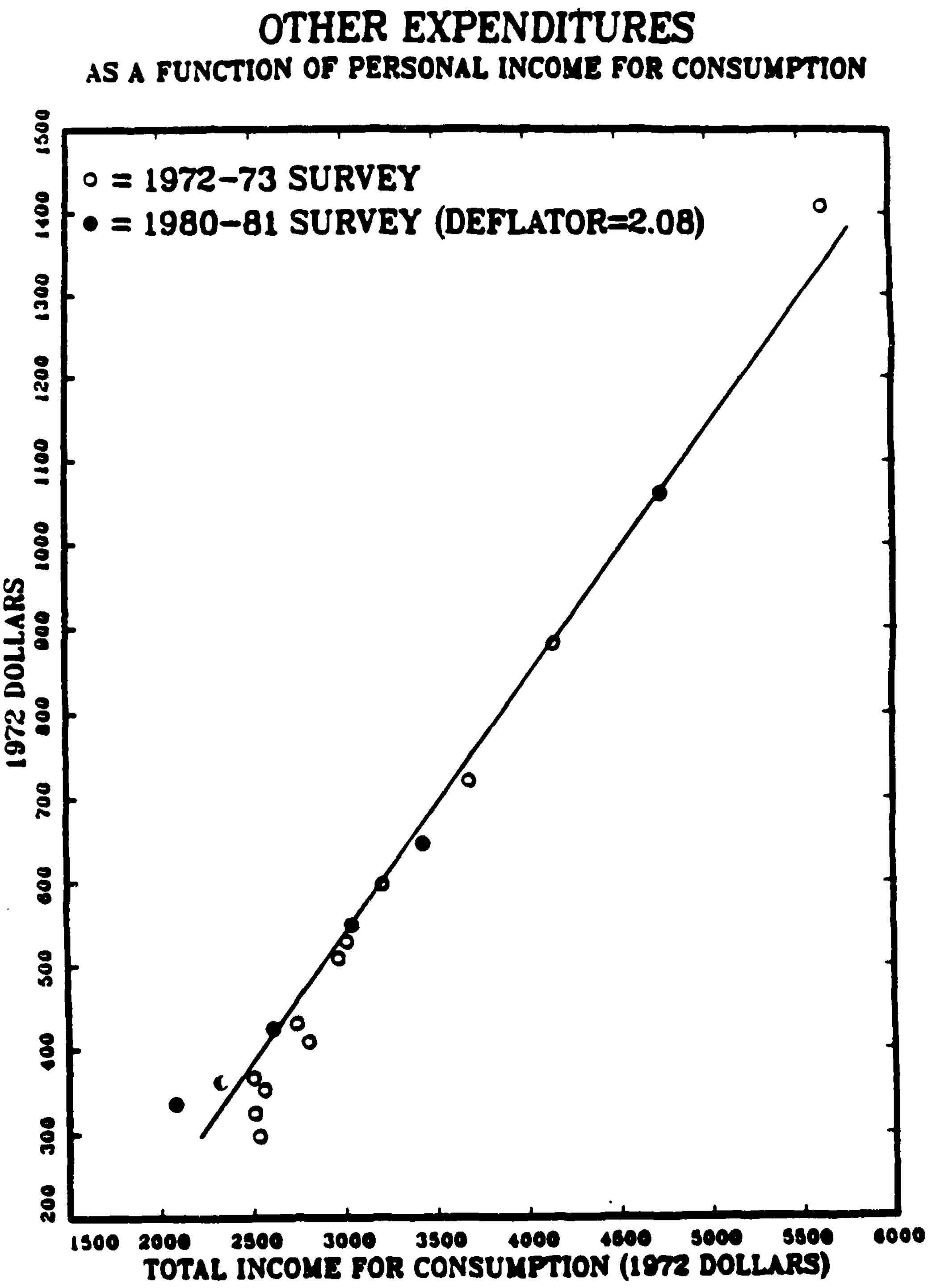

OTHER EXPENDITURES
AS A FUNCTION OF PERSONAL INCOME FOR CONSUMPTION
○ = 1972-73 SURVEY
● = 1980-81 SURVEY (DEFLATOR=2.08)
1972 DOLLARS
200
300
400
500
600
700
800
900
1000
1100
1200
1300
1400
1500
1500
2000
2500
3000
3500
4000
4500
5000
5500
6000
TOTAL INCOME FOR CONSUMPTION (1972 DOLLARS)

Table 3

**$\tilde{X}_t$ OBSERVED CONSUMPTION PER CAPITA BY YEARS (1972$)
BY NIPA CATEGORIES**

YEAR	FOOD	CLOTH	HOUSING	HOUSE OPER.	TRANSP.	RECREATION	PER. CARE	OTHER	TOTAL
1961	700	203	448	216	327	178	67	429	2568
1962	699	208	465	224	337	186	71	432	2622
1963	699	209	480	231	347	194	71	446	2677
1964	712	222	496	246	360	203	74	470	2783
1965	732	228	516	255	378	217	77	489	2892
1966	743	239	533	271	398	244	80	506	3014
1967	748	236	550	275	413	254	82	522	3080
1968	768	244	570	282	440	268	84	544	3200
1969	775	247	592	286	467	279	83	567	3296
1970	789	241	605	280	478	289	83	586	3351
1971	785	249	621	279	503	293	80	603	3413
*1972	796	264	649	297	541	315	82	622	3566
1973	785	280	675	317	568	342	82	635	3684
1974	774	274	692	305	557	350	78	635	3665
1975	787	282	709	285	564	362	73	648	3710
1976	818	293	738	291	600	384	72	671	3867
1977	838	304	766	306	631	406	72	698	4021
1978	836	329	803	317	662	433	74	712	4166
1979	843	341	813	328	655	450	74	735	4239
1980	854	342	826	321	611	452	73	745	4224
1981	852	362	848	318	605	473	71	762	4291
1982	852	364	860	302	606	474	68	778	4304

*base year

SOURCE: National Income and Product Accounts, special supplement to the Survey of Current Business Statistic Tables published September 1981 for years 1929–76, 1976–79 and Revised Estimates published July 1983/Vol. 63, No. 7. Table 2.4 Personal Consumption Expenditures by Type of Expenditure.

Table 4

p_t OBSERVED PRICES BY YEARS RELATIVE TO BASE YEAR 1972 BY NIPA CATEGORIES

YEAR	FOOD	CLOTH	HOUSING	HOUSE OPER.	TRANSP.	RECREA-TION	PER. CARE	OTHER	$\overline{e\pi}$
1961	.97	1.01	1.07	1.05	1.08	1.07	1.03	.86	1.03
1962	.96	1.00	1.06	1.04	1.08	1.07	1.02	.88	1.02
1963	.96	1.00	1.06	1.06	1.04	1.07	1.03	.91	1.02
1964	.97	.99	1.05	1.04	1.06	1.07	1.02	.89	1.02
1965	.97	.98	1.04	1.03	1.06	1.07	1.03	.92	1.01
1966	.99	.98	1.03	1.03	1.05	1.05	1.03	.92	1.01
1967	.98	1.00	1.02	1.02	1.04	1.04	1.03	.93	1.02
1968	.98	1.02	1.01	1.03	1.03	1.04	1.04	.96	1.01
1969	.99	1.03	1.00	1.02	1.02	1.03	1.02	.97	1.01
1970	1.00	1.02	.99	1.02	1.01	1.01	1.02	.98	1.01
1971	.99	1.01	1.00	1.01	1.02	1.00	1.01	.97	1.00
*1972	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1973	1.05	.98	1.00	.97	.97	.97	.99	1.01	1.00
1974	1.08	.95	.98	.98	.99	.93	.99	.99	1.00
1975	1.07	.91	.98	1.01	.99	.91	1.03	1.00	1.00
1976	1.05	.90	.99	1.03	1.01	.90	1.05	1.03	.99
1977	1.04	.88	1.00	1.00	1.03	.88	1.06	1.04	.99
1978	1.06	.84	1.00	1.01	1.02	.85	1.07	1.06	.99
1979	1.07	.80	1.01	.99	1.06	.81	1.07	1.07	.98
1980	1.05	.75	1.02	.97	1.12	.79	1.06	1.07	.97
1981	1.05	.71	1.03	.97	1.13	.77	1.06	1.07	.97
1982	1.04	.69	1.05	.99	1.09	.77	1.10	1.08	.97

* base year

SOURCE: National Income and Product Accounts, special supplement to the Survey of Current Business Statistic Tables published September 1981 for years 1929-76, 1976-79 and Revised Estimates published July 1983/Vol. 63, No. 7. Table 2.4 Personal Consumption Expenditures by Type of Expenditure. Price index is derived by comparing expenditures in current dollars in Table 2.5 with those in Table 2.4.

$\overline{e\pi}_t$ = average value of components of $\bar{\pi}_t = p_t/(p_t'Hp_t)^{1/2}$

Table 5

$\bar{X}_t$ PREDICTED CONSUMPTION PER CAPITA BY YEARS (1972$)
BY NIPA CATEGORIES BASED ON DEMAND FUNCTION

YEAR	FOOD	CLOTH	HOUSING	HOUSE OPER.	TRANSP.	RECREATION	PER. CARE	OTHER
1961	703	177	431	228	322	189	68	454
1962	713	194	454	219	327	185	89	443
1963	723	200	466	225	351	196	85	430
1964	718	226	498	227	362	181	97	477
1965	728	237	534	255	382	194	77	485
1966	731	242	558	267	408	219	69	523
1967	744	228	562	295	425	245	47	536
1968	763	236	587	293	456	271	52	543
1969	772	242	601	288	475	287	75	556
1970	775	249	615	281	492	304	78	558
1971	785	246	616	286	493	315	86	585
1972	798	271	645	285	534	325	110	598
1973	783	273	663	337	578	342	77	633
1974	767	282	682	312	556	346	81	641
1975	778	299	705	285	573	363	71	640
1976	812	314	735	260	595	397	96	659
1977	835	317	763	291	610	423	80	706
1978	839	334	799	312	645	442	64	723
1979	845	335	821	334	636	459	54	757
1980	852	344	838	329	592	444	72	754
1981	859	359	850	311	599	451	90	766
1982	864	357	837	315	643	469	57	760
$\bar{S}$	2666	1970	4670	2050	3479	2253	526	7340
Avg. % |error|	.9	3.4	1.7	3.4	2.1	3.8	17.1	2.2

% |error| = 100 $|\bar{X}_{tk} - \tilde{X}_{tk}|/\bar{X}_{tk}$, $\tilde{X}_{tk}$ are tabulated in Table 3.

$\bar{X}_t$ computed using per capita demand function given in Table 1.

Graphs that follow this Table comparing $\bar{X}$ with $\tilde{X}$ are by Isabel Pereira.

Figure 9

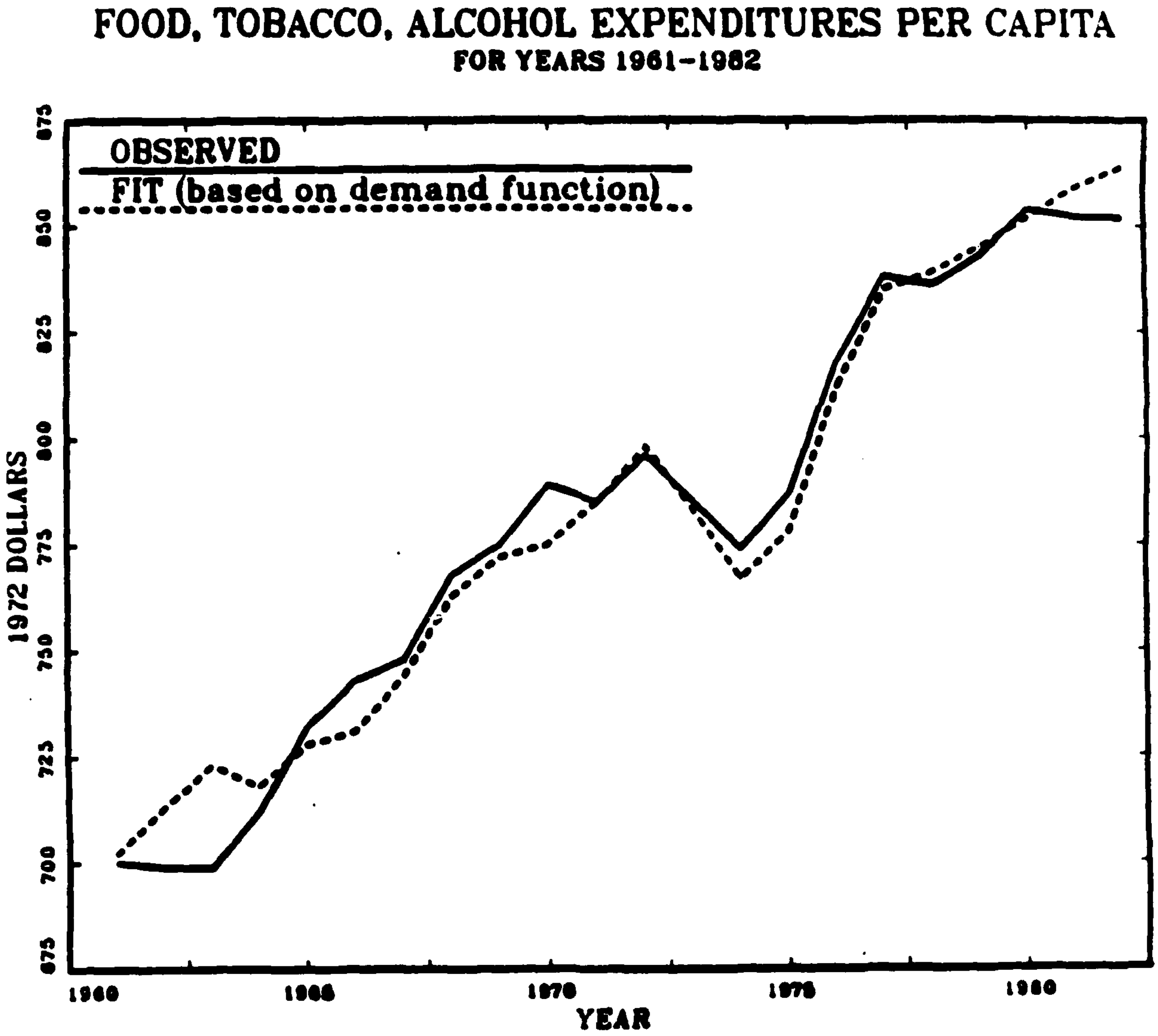

FOOD, TOBACCO, ALCOHOL EXPENDITURES PER CAPITA
FOR YEARS 1961-1982
OBSERVED
FIT (based on demand function)
1972 DOLLARS
YEAR
675
700
725
750
775
800
825
850
875
1960
1965
1970
1975
1980

Figure 10

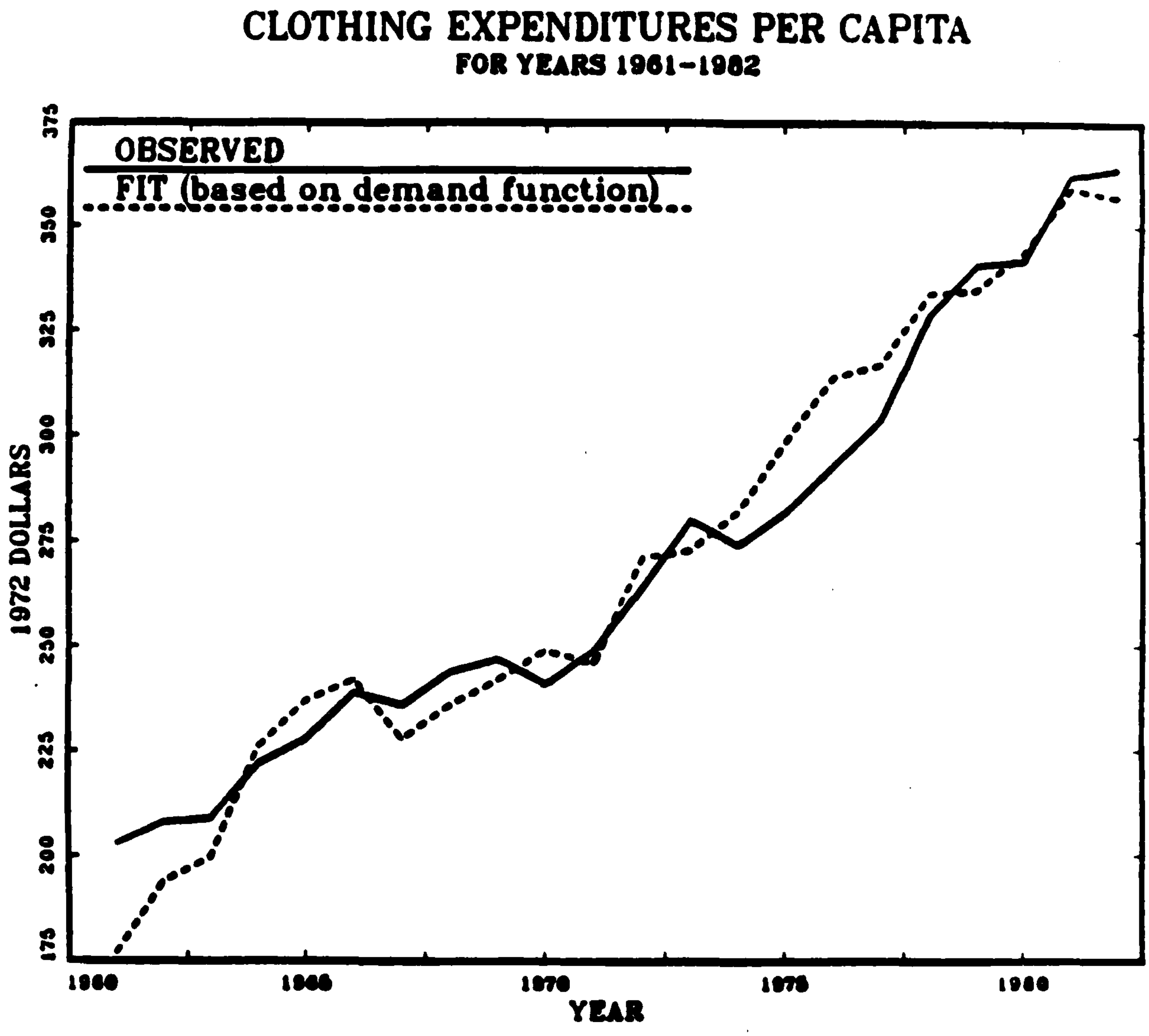

CLOTHING EXPENDITURES PER CAPITA
FOR YEARS 1961-1982
OBSERVED
FIT (based on demand function)
1972 DOLLARS
175
200
225
250
275
300
325
350
375
1960
1965
1970
1975
1980
YEAR

Figure 11

HOUSING EXPENDITURES PER CAPITA
FOR YEARS 1961-1982

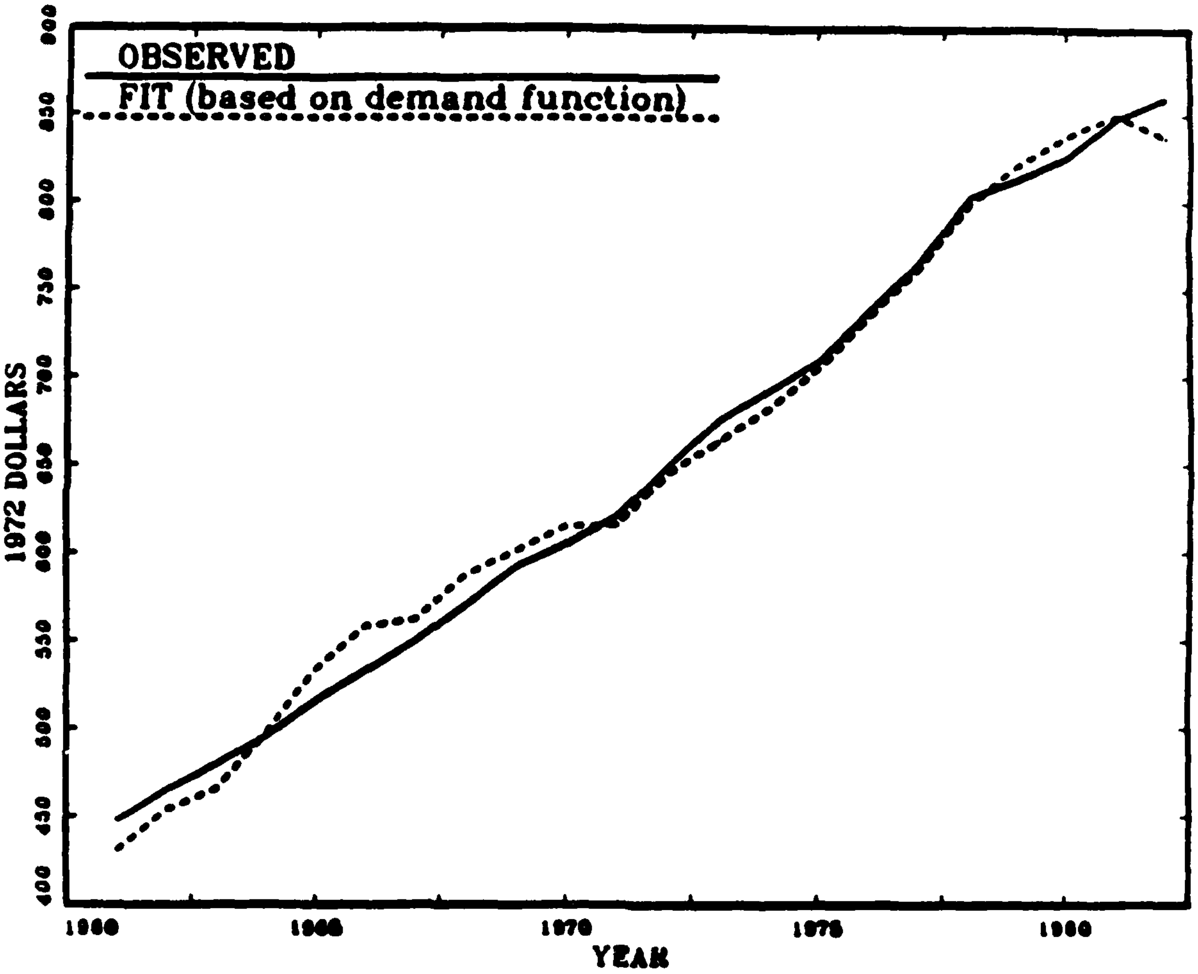


Figure 12

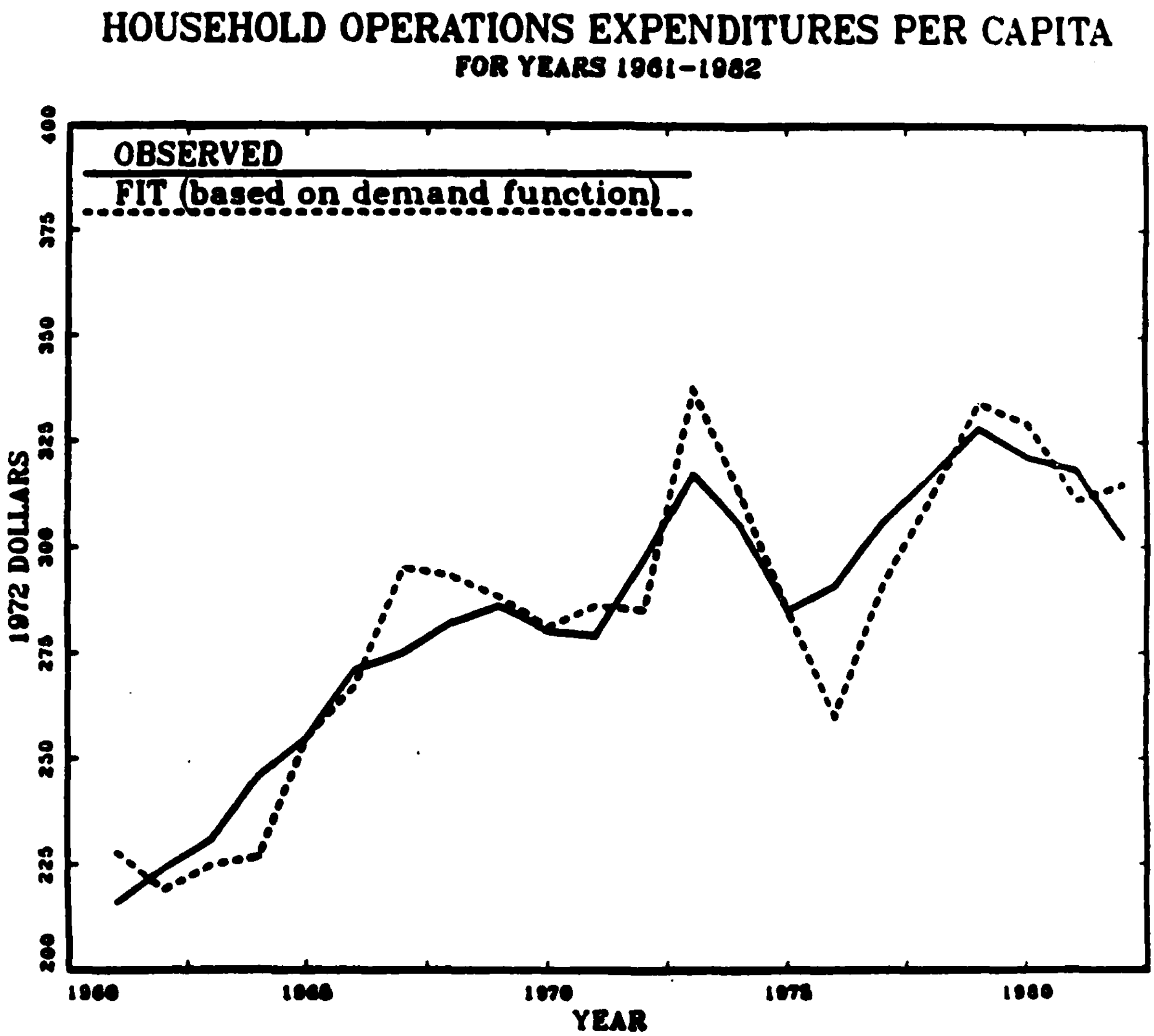

HOUSEHOLD OPERATIONS EXPENDITURES PER CAPITA
FOR YEARS 1961-1982
OBSERVED
FIT (based on demand function)
1972 DOLLARS
YEAR

Figure 13

TRANSPORTATION EXPENDITURES PER CAPITA
FOR YEARS 1961–1982

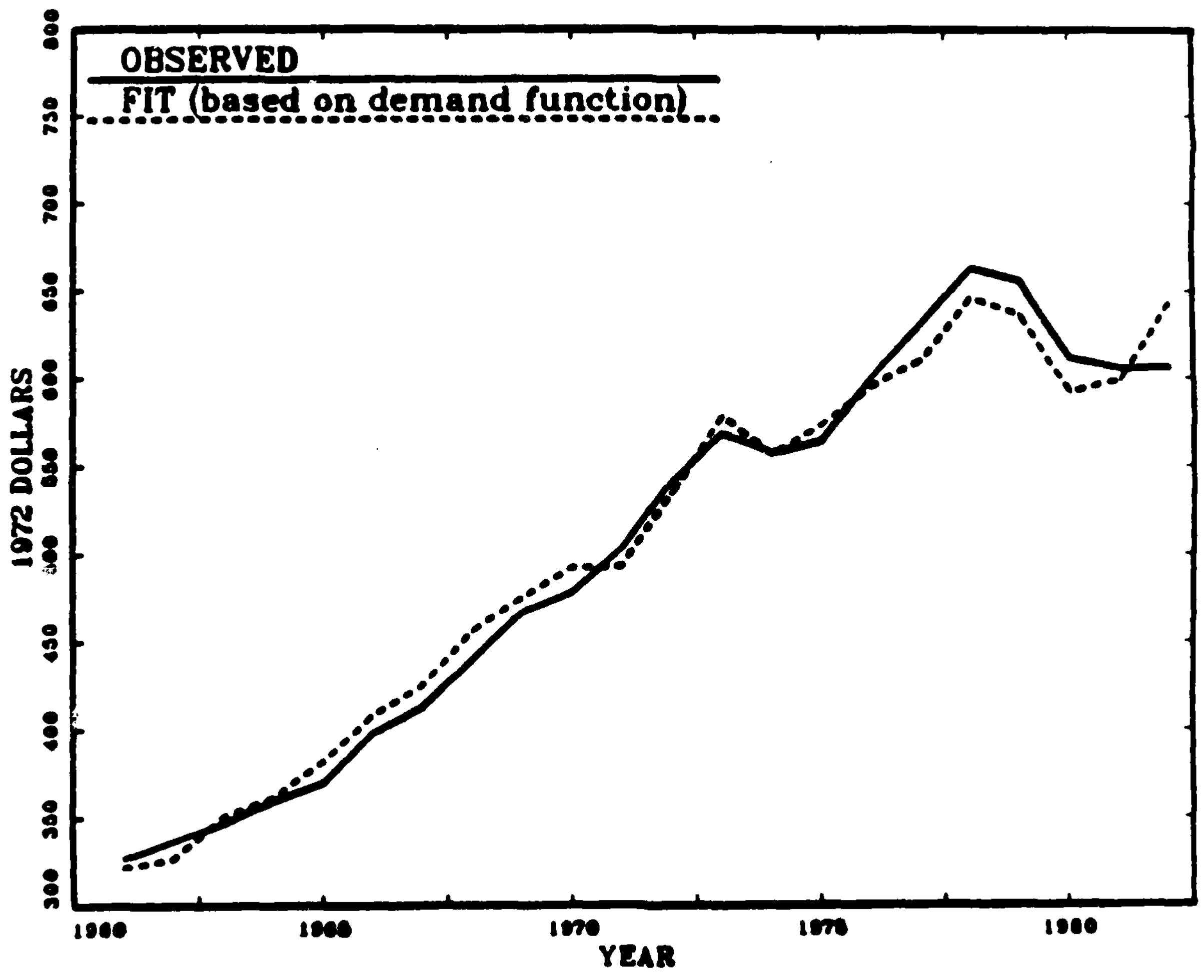


Figure 14

RECREATION EXPENDITURES PER CAPITA
FOR YEARS 1961-1982

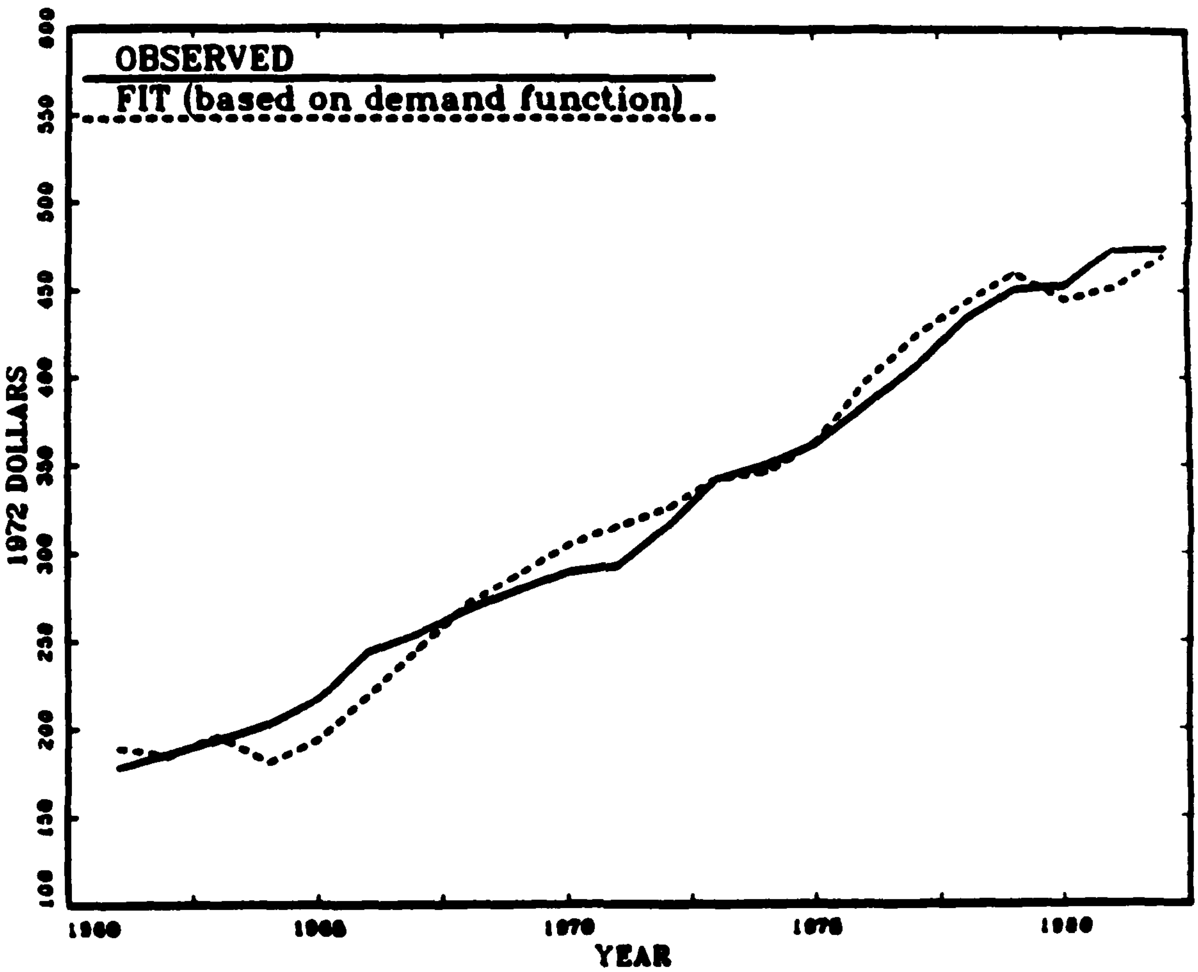
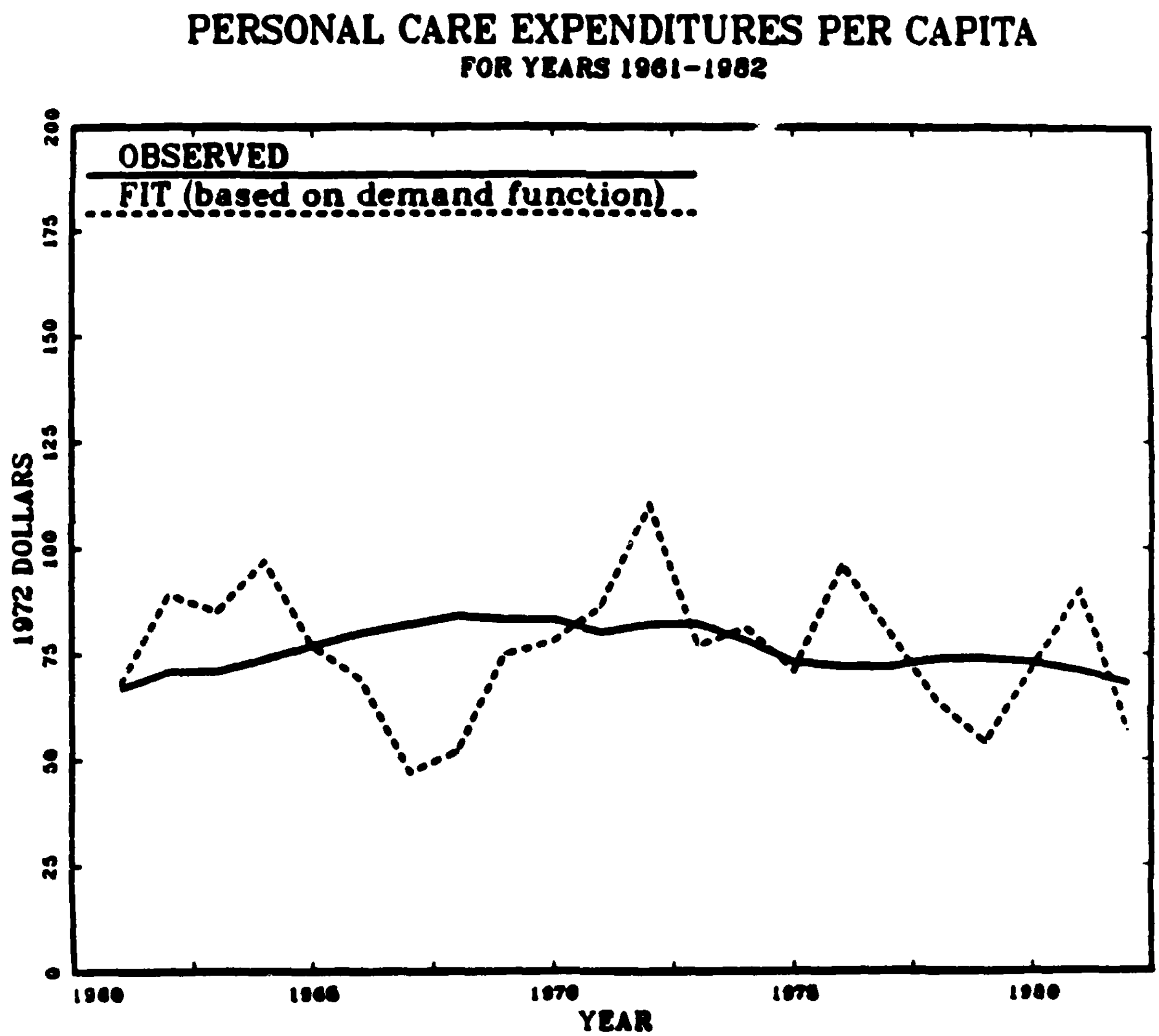
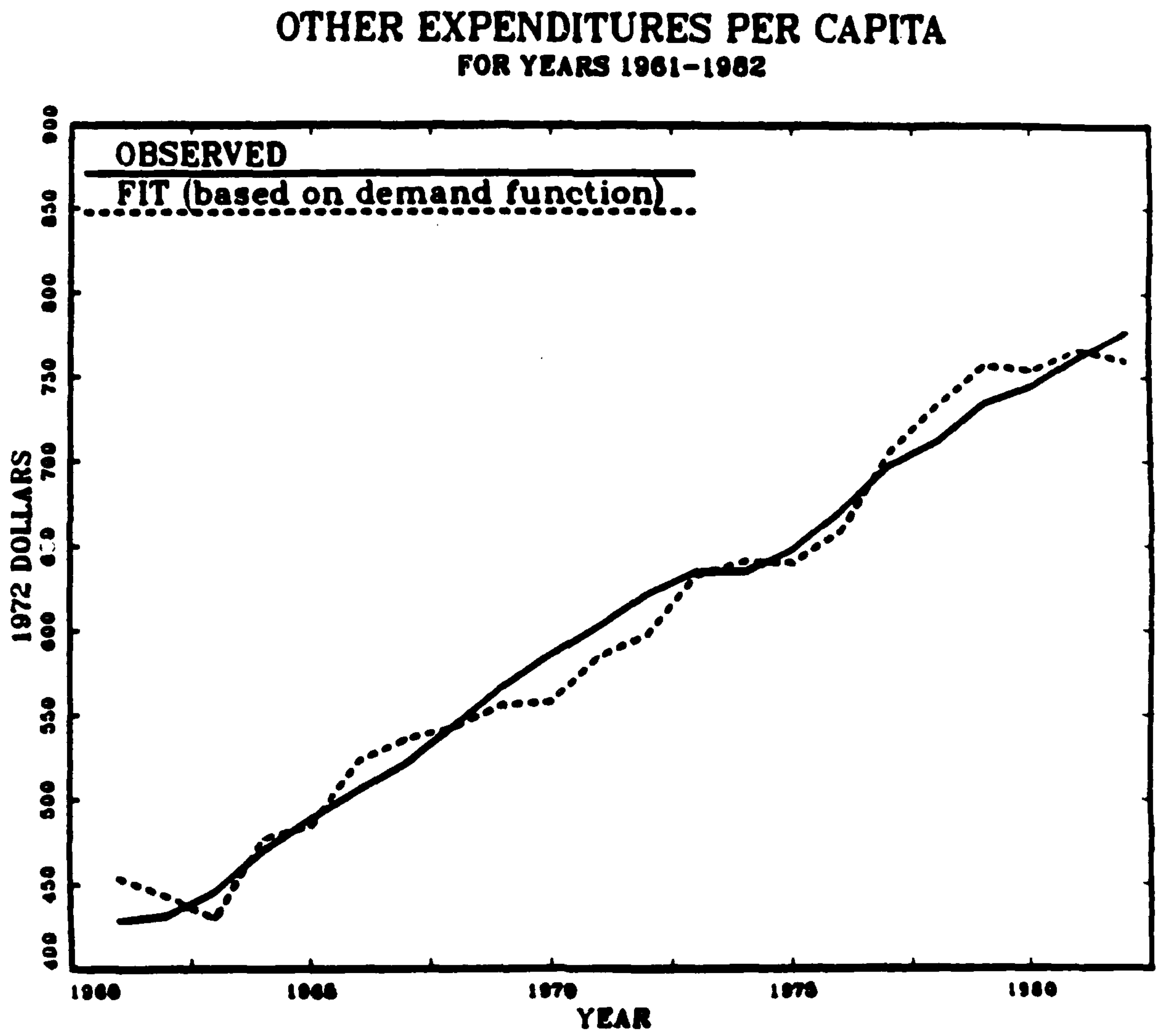


Figure 15



PERSONAL CARE EXPENDITURES PER CAPITA
FOR YEARS 1961-1982
OBSERVED
FIT (based on demand function)
1972 DOLLARS
YEAR

Figure 16



OTHER EXPENDITURES PER CAPITA
FOR YEARS 1961-1982
OBSERVED
FIT (based on demand function)
1972 DOLLARS
YEAR

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SOL 88-6: Deriving a Utility Function for the U.S. Economy, by G.B. Dantzig, J.C. Stone and P.H. McAllister.

ABSTRACT

Given a general dynamic equilibrium formulation of a time staged model, we seek conditions on the distribution of utility functions of individuals which imply the model is equivalent to a mathematical program.

Gorman and others long ago have observed that Engel curves of average consumption as a function of income at fixed prices are remarkably linear over a broad range of income of interest which tapers off at both ends of this range. We reproduce this phenomenon by assuming (a) that a general polynomial of the second degree has enough parameters (coefficients) to globally represent the utility functions of individual consumers, and (b) the distribution of utility functions that individuals have is independent of the income they happen to have. We achieve the latter by assigning values to the parameters of the utility functions by a random drawing with replacement from a "population urn" containing a representative sets of the parameters. We then derive the functional form of the per capita demand function and necessary and sufficient conditions for its integrability.

Finally, we show in the context of the time staged model, that when the population is not too polarized as to its tastes at fixed income levels, a concave objective function always exists, which maximized subject to the physical flow constraints, implies the equilibrium conditions.